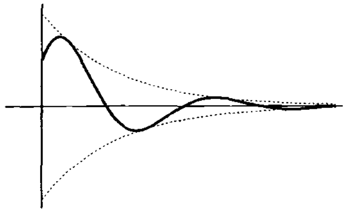

B

Background

The topics discussed in this chapter are not entirely new to students taking this course. You have already studied many of these topics in earlier courses or are expected to know them from your previous training. Even so, this background material deserves a review because it is so pervasive in the area of signals and systems. Investing a little time in such a review will pay big dividends later. Furthermore, this material is useful not only for this course but also for several courses that follow. It will also be helpful as reference material in your future professional career.

B.1 Complex Numbers

Complex numbers are an extension of ordinary numbers and are an integral part of the modern number system. Complex numbers, particularly **imaginary numbers**, sometimes seem mysterious and unreal. This feeling of unreality derives from their unfamiliarity and novelty rather than their supposed nonexistence! Mathematicians blundered in calling these numbers "imaginary," for the term immediately prejudices perception. Had these numbers been called by some other name, they would have become demystified long ago, just as irrational numbers or negative numbers were. Many futile attempts have been made to ascribe some physical meaning to imaginary numbers. However, this effort is needless. In mathematics we assign symbols and operations any meaning we wish as long as internal consistency is maintained. A healthier approach would have been to define a symbol i (with any term but "imaginary"), which has a property $i^2 = -1$. The history of mathematics is full of entities which were unfamiliar and held in abhorrence until familiarity made them acceptable. This fact will become clear from the following historical note.

B.1-1 A Historical Note

Among early people the number system consisted only of natural numbers (positive integers) needed to count the number of children, cattle, and quivers of arrows. These people had no need for fractions. Whoever heard of two and one-half children or three and one-fourth cows!

However, with the advent of agriculture, people needed to measure continuously varying quantities, such as the length of a field, the weight of a quantity of butter, and so on. The number system, therefore, was extended to include fractions. The ancient Egyptians and Babylonians knew how to handle fractions, but **Pythagoras** discovered that some numbers (like the diagonal of a unit square) could not be expressed as a whole number or a fraction. Pythagoras, a number mystic, who regarded numbers as the essence and principle of all things in the universe, was so appalled at his discovery that he swore his followers to secrecy and imposed a death penalty for divulging this secret.[1] These numbers, however, were included in the number system by the time of Descartes, and they are now known as **irrational numbers**.

Until recently, **negative numbers** were not a part of the number system. The concept of negative numbers must have appeared absurd to early man. However, the medieval Hindus had a clear understanding of the significance of positive and negative numbers.[2,3] They were also the first to recognize the existence of absolute negative quantities.[4] The works of **Bhaskar** (1114-1185) on arithmetic (*Līlāvatī*) and algebra (*Bījaganit*) not only use the decimal system but also give rules for dealing with negative quantities. Bhaskar recognized that positive numbers have two square roots.[5] Much later, in Europe, the banking system that arose in Florence and Venice during the late Renaissance (fifteenth century) is credited with developing a crude form of negative numbers. The seemingly absurd subtraction of 7 from 5 seemed reasonable when bankers began to allow their clients to draw seven gold ducats while their deposit stood at five. All that was necessary for this purpose was to write the difference, 2, on the debit side of a ledger.[6]

Thus the number system was once again broadened (generalized) to include negative numbers. The acceptance of negative numbers made it possible to solve equations such as $x+5=0$, which had no solution before. Yet for equations such as $x^2+1=0$, leading to $x^2=-1$, the solution could not be found in the real number system. It was therefore necessary to define a completely new kind of number with its square equal to -1. During the time of Descartes and Newton, imaginary (or complex) numbers came to be accepted as part of the number system, but they were still regarded as algebraic fiction. The Swiss mathematician **Leonhard Euler** introduced the notation i (for **imaginary**) around 1777 to represent $\sqrt{-1}$. Electrical engineers use the notation j instead of i to avoid confusion with the notation i often used for electrical current. Thus

$$j^2=-1 \quad .$$

and

$$\sqrt{-1}=\pm j$$

This notation allows us to determine the square root of any negative number. For example,

$$\sqrt{-4}=\sqrt{4}\times\sqrt{-1}=\pm 2j$$

When imaginary numbers are included in the number system, the resulting numbers are called **complex numbers**.

Origins of Complex Numbers

Ironically (and contrary to popular belief), it was not the solution of a quadratic equation, such as $x^2+1=0$, but a cubic equation with real roots that made

Gerolamo Cardano (left) and Karl Friedrich Gauss (right).

imaginary numbers plausible and acceptable to early mathematicians. They could dismiss $\sqrt{-1}$ as pure nonsense when it appeared as a solution to $x^2 + 1 = 0$ because this equation has no real solution. But in 1545, **Gerolamo Cardano** of Milan published *Ars Magna* (*The Great Art*), the most important algebraic work of the Renaissance. In this book he gave a method of solving a general cubic equation in which a root of a negative number appeared in an intermediate step. According to his method, the solution to a third-order equation†

$$x^3 + ax + b = 0$$

is given by

$$x = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} + \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$$

For example, to find a solution of $x^3 + 6x - 20 = 0$, we substitute $a = 6$, $b = -20$ in the above equation to obtain

$$x = \sqrt[3]{10 + \sqrt{108}} + \sqrt[3]{10 - \sqrt{108}} = \sqrt[3]{20.392} - \sqrt[3]{0.392} = 2$$

We can readily verify that 2 is indeed a solution of $x^3 + 6x - 20 = 0$. But when Cardano tried to solve the equation $x^3 - 15x - 4 = 0$ by this formula, his solution

†This equation is known as the *depressed cubic* equation. A general cubic equation

$$y^3 + py^2 + qy + r = 0$$

can always be reduced to a depressed cubic form by substituting $y = x - \frac{p}{3}$. Therefore any general cubic equation can be solved if we know the solution to the depressed cubic. The depressed cubic was independently solved, first by **Scipione del Ferro** (1465-1526) and then by **Niccolo Fontana** (1499-1557). The latter is better known in the history of mathematics as **Tartaglia** ("Stammerer"). Cardano learned the secret of the depressed cubic solution from Tartaglia. He then showed that by using the substitution $y = x - \frac{p}{3}$, a general cubic is reduced to a depressed cubic.

was

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

What was Cardano to make of this equation in the year 1545? In those days negative numbers were themselves suspect, and a square root of a negative number was doubly preposterous! Today we know that

$$(2 \pm j)^3 = 2 \pm j11 = 2 \pm \sqrt{-121}$$

Therefore, Cardano's formula gives

$$x = (2 + j) + (2 - j) = 4$$

We can readily verify that $x = 4$ is indeed a solution of $x^3 - 15x - 4 = 0$. Cardano tried to explain halfheartedly the presence of $\sqrt{-121}$ but ultimately dismissed the whole enterprise as being "as subtle as it is useless." A generation later, however, **Raphael Bombelli** (1526-1573), after examining Cardano's results, proposed acceptance of imaginary numbers as a necessary vehicle that would transport the mathematician from the *real* cubic equation to its *real* solution. In other words, while we begin and end with real numbers, we seem compelled to move into an unfamiliar world of imaginaries to complete our journey. To mathematicians of the day, this proposal seemed incredibly strange.[7] Yet they could not dismiss the idea of imaginary numbers so easily because this concept yielded the real solution of an equation. It took two more centuries for the full importance of complex numbers to become evident in the works of Euler, Gauss, and Cauchy. Still, Bombelli deserves credit for recognizing that such numbers have a role to play in algebra.[7]

In 1799, the German mathematician **Karl Friedrich Gauss**, at a ripe age of 22, proved the fundamental theorem of algebra, namely that every algebraic equation in one unknown has a root in the form of a complex number. He showed that every equation of the nth order has exactly n solutions (roots), no more and no less. Gauss was also one of the first to give a coherent account of complex numbers and to interpret them as points in a complex plane. It is he who introduced the term *complex numbers* and paved the way for general and systematic use of complex numbers. The number system was once again broadened or generalized to include imaginary numbers. Ordinary (or real) numbers became a special case of generalized (or complex) numbers.

The utility of complex numbers can be understood readily by an analogy with two neighboring countries X and Y, as illustrated in Fig. B.1. If we want to travel from City a to City b (both in Country X), the shortest route is through Country Y, although the journey begins and ends in Country X. We may, if we desire, perform this journey by an alternate route that lies exclusively in X, but this alternate route is longer. In mathematics we have a similar situation with real numbers (Country X) and complex numbers (Country Y). All real-world problems must start with real numbers, and all the final results must also be in real numbers. But the derivation of results is considerably simplified by using complex numbers as an intermediary. It is also possible to solve all real-world problems by an alternate method, using real numbers exclusively, but such procedure would increase the work needlessly.

B.1-2 Algebra of Complex Numbers

A complex number $(a,\, b)$ or $a + jb$ can be represented graphically by a point whose Cartesian coordinates are $(a,\, b)$ in a complex plane (Fig. B.2). Let us denote this complex number by z so that

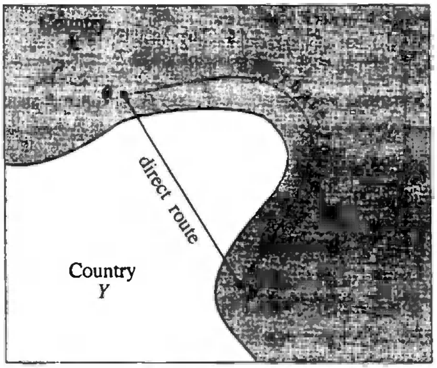

Fig. B.1 Use of complex numbers can reduce the work.

$$z = a + jb \tag{B.1}$$

The numbers a and b (the abscissa and the ordinate) of z are the **real part** and the **imaginary part**, respectively, of z. They are also expressed as

$$\text{Re } z = a$$

$$\text{Im } z = b$$

Note that in this plane all real numbers lie on the horizontal axis, and all imaginary numbers lie on the vertical axis.

Complex numbers may also be expressed in terms of polar coordinates. If (r, θ) are the polar coordinates of a point $z = a + jb$ (see Fig. B.2), then

$$a = r \cos \theta$$

$$b = r \sin \theta$$

and

$$\begin{aligned} z = a + jb &= r \cos \theta + jr \sin \theta \\ &= r(\cos \theta + j \sin \theta) \end{aligned} \tag{B.2}$$

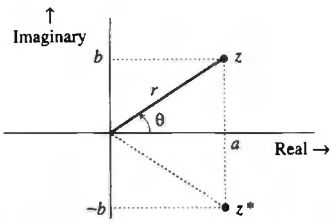

Fig. B.2 Representation of a number in the complex plane.

The **Euler formula** states that

$$e^{j\theta} = \cos\theta + j\sin\theta$$

To prove the Euler formula, we expand $e^{j\theta}$, $\cos\theta$, and $\sin\theta$ using a Maclaurin series

$$\begin{aligned}
e^{j\theta} &= 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \frac{(j\theta)^6}{6!} + \cdots \\
&= 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - \cdots \\
\cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!}\cdots \\
\sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots
\end{aligned}$$

Hence, it follows that

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{B.3}$$

Using (B.3) in (B.2) yields

$$\begin{aligned}
z &= a + jb \\
&= re^{j\theta}
\end{aligned} \tag{B.4}$$

Thus, a complex number can be expressed in Cartesian form $a + jb$ or polar form $re^{j\theta}$ with

$$a = r\cos\theta, \qquad b = r\sin\theta \tag{B.5}$$

and

$$r = \sqrt{a^2 + b^2}, \qquad \theta = \tan^{-1}\left(\frac{b}{a}\right) \tag{B.6}$$

Observe that r is the distance of the point z from the origin. For this reason, r is also called the **magnitude** (or **absolute value**) of z and is denoted by $|z|$. Similarly θ is called the angle of z and is denoted by $\angle z$. Therefore

$$|z| = r, \qquad \angle z = \theta$$

and

$$z = |z|e^{j\angle z} \tag{B.7}$$

Also

$$\frac{1}{z} = \frac{1}{re^{j\theta}} = \frac{1}{r}e^{-j\theta} = \frac{1}{|z|}e^{-j\angle z} \tag{B.8}$$

Conjugate of a Complex Number

We define z^*, the **conjugate** of $z = a + jb$, as

$$z^* = a - jb = re^{-j\theta} \tag{B.9a}$$

$$= |z|e^{-j\angle z} \tag{B.9b}$$

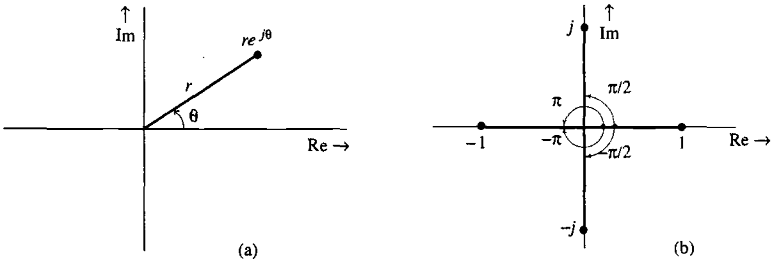

Fig. B.3 Understanding some useful identities in terms of $re^{j\theta}$.

The graphical representation of a number z and its conjugate z^* is depicted in Fig. B.2. Observe that z^* is a mirror image of z about the horizontal axis. *To find the conjugate of any number, we need only to replace j by $-j$ in that number* (which is the same as changing the sign of its angle).

The sum of a complex number and its conjugate is a real number equal to twice the real part of the number:

$$z + z^* = (a + jb) + (a - jb) = 2a = 2\,\mathrm{Re}\,z \tag{B.10a}$$

The product of a complex number z and its conjugate is a real number $|z|^2$, the square of the magnitude of the number:

$$zz^* = (a + jb)(a - jb) = a^2 + b^2 = |z|^2 \tag{B.10b}$$

Understanding Some Useful Identities

In a complex plane, $re^{j\theta}$ represents a point at a distance r from the origin and at an angle θ with the horizontal axis, as shown in Fig. B.3a. For example, the number -1 is at a unit distance from the origin and has an angle π or $-\pi$ (in fact, any odd multiple of $\pm\pi$), as seen from Fig. B.3b. Therefore,

$$1e^{\pm j\pi} = -1$$

In fact,

$$e^{\pm jn\pi} = -1 \qquad n \text{ odd integer} \tag{B.11}$$

The number 1, on the other hand, is also at a unit distance from the origin, but has an angle 2π (in fact, $\pm 2n\pi$ for any integral value of n). Therefore,

$$e^{\pm j2n\pi} = 1 \qquad n \text{ integer} \tag{B.12}$$

The number j is at unit distance from the origin and its angle is $\pi/2$ (see Fig. B.3b). Therefore,

$$e^{j\pi/2} = j$$

Similarly,

$$e^{-j\pi/2} = -j$$

Thus

$$e^{\pm j\pi/2} = \pm j \tag{B.13a}$$

In fact,

$$e^{\pm jn\pi/2} = \pm j \qquad n = 1,\ 5,\ 9,\ 13, \cdots \tag{B.13b}$$

and

$$e^{\pm jn\pi/2} = \mp j \qquad n = 3,\ 7,\ 11,\ 15, \cdots \tag{B.13c}$$

These results are summarized in Table B.1.

TABLE B.1

r	θ	$re^{j\theta}$	
1	0	$e^{j0} = 1$	
1	$\pm\pi$	$e^{\pm j\pi} = -1$	
1	$\pm n\pi$	$e^{\pm jn\pi} = -1$	n odd integer
1	$\pm 2\pi$	$e^{\pm j2\pi} = 1$	
1	$\pm 2n\pi$	$e^{\pm j2n\pi} = 1$	n integer
1	$\pm\pi/2$	$e^{\pm j\pi/2} = \pm j$	
1	$\pm n\pi/2$	$e^{\pm jn\pi/2} = \pm j$	$n = 1, 5, 9, 13, \ldots$
1	$\pm n\pi/2$	$e^{\pm jn\pi/2} = \mp j$	$n = 3, 7, 11, 15, \ldots$

This discussion shows the usefulness of the graphic picture of $re^{j\theta}$. This picture is also helpful in several other applications. For example, to determine the limit of $e^{(\alpha+j\omega)t}$ as $t \to \infty$, we note that

$$e^{(\alpha+j\omega)t} = e^{\alpha t} e^{j\omega t}$$

Now the magnitude of $e^{j\omega t}$ is unity regardless of the value of ω or t because $e^{j\omega t} = re^{j\theta}$ with $r = 1$. Therefore, $e^{\alpha t}$ determines the behavior of $e^{(\alpha+j\omega)t}$ as $t \to \infty$ and

$$\lim_{t\to\infty} e^{(\alpha+j\omega)t} = \lim_{t\to\infty} e^{\alpha t} e^{j\omega t} = \begin{cases} 0 & \alpha < 0 \\ \infty & \alpha > 0 \end{cases} \tag{B.14}$$

In future discussions you will find it very useful to remember $re^{j\theta}$ as a number at a distance r from the origin and at an angle θ with the horizontal axis of the complex plane.

A Warning About Using Electronic Calculators in Computing Angles

From the Cartesian form $a + jb$ we can readily compute the polar form $re^{j\theta}$ [see Eq. (B.6)]. Electronic calculators provide ready conversion of rectangular into polar and vice versa. However, if a calculator computes an angle of a complex number using an inverse trigonometric function $\theta = \tan^{-1}(b/a)$, proper attention must be paid to the quadrant in which the number is located. For instance, θ corresponding to the number $-2 - j3$ is $\tan^{-1}(\frac{-3}{-2})$. This result is not the same as $\tan^{-1}(\frac{3}{2})$. The former is $-123.7°$, whereas the latter is $56.3°$. An electronic calculator cannot make this distinction and can give a correct answer only for angles in the first and

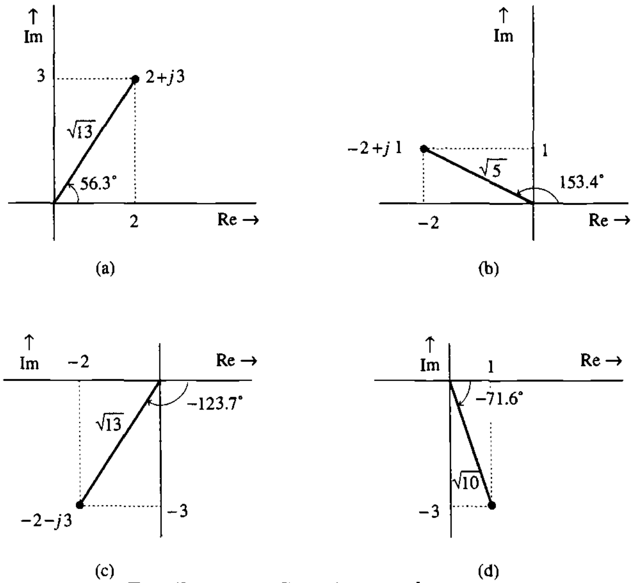

Fig. B.4 From Cartesian to polar form.

fourth quadrants. It will read $\tan^{-1}(\frac{-3}{-2})$ as $\tan^{-1}(\frac{3}{2})$, which is clearly wrong. In computing inverse trigonometric functions, if the angle appears in the second or third quadrant, the answer of the calculator is off by 180°. The correct answer is obtained by adding or subtracting 180° to the value found with the calculator (either adding or subtracting yields the correct answer). For this reason it is advisable to draw the point in the complex plane and determine the quadrant in which the point lies. This issue will be clarified by the following examples.

■ **Example B.1**

Express the following numbers in polar form:

(a) $2 + j3$ **(b)** $-2 + j1$ **(c)** $-2 - j3$ **(d)** $1 - j3$

(a)

$$|z| = \sqrt{2^2 + 3^2} = \sqrt{13} \qquad \angle z = \tan^{-1}\left(\tfrac{3}{2}\right) = 56.3^\circ$$

In this case the number is in the first quadrant, and a calculator will give the correct value of 56.3°. Therefore, (see Fig. B.4a)

$$2 + j3 = \sqrt{13}\, e^{j56.3^\circ}$$

(b)

$$|z| = \sqrt{(-2)^2 + 1^2} = \sqrt{5} \qquad \angle z = \tan^{-1}\left(\tfrac{1}{-2}\right) = 153.4^\circ$$

In this case the angle is in the second quadrant (see Fig. B.4b), and therefore the answer given by the calculator ($\tan^{-1}(\frac{1}{-2}) = -26.6^\circ$) is off by 180°. The correct answer is $(-26.6 \pm 180)^\circ = 153.4^\circ$ or -206.6°. Both values are correct because they represent the same angle. As a matter of convenience, we choose an angle whose numerical value is less than 180°, which in this case is 153.4°. Therefore,

$$-2 + j1 = \sqrt{5} e^{j153.4^\circ}$$

(c)
$$|z| = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13} \qquad \angle z = \tan^{-1}(\tfrac{-3}{-2}) = -123.7^\circ$$

In this case the angle appears in the third quadrant (see Fig. B.4c), and therefore the answer obtained by the calculator ($\tan^{-1}(\frac{-3}{-2}) = 56.3^\circ$) is off by 180°. The correct answer is $(56.3 \pm 180)^\circ = 236.3^\circ$ or -123.7°. As a matter of convenience, we choose the latter and (see Fig. B.4c)

$$-2 - j3 = \sqrt{13}e^{-j123.7^\circ}$$

(d)
$$|z| = \sqrt{1^2 + (-3)^2} = \sqrt{10} \qquad \angle z = \tan^{-1}(\tfrac{-3}{1}) = -71.6^\circ$$

In this case the angle appears in the fourth quadrant (see Fig. B.4d), and therefore the answer given by the calculator ($\tan^{-1}(\frac{-3}{1}) = -71.6^\circ$) is correct (see Fig. B.4d).

$$1 - j3 = \sqrt{10}e^{-j71.6^\circ} \quad \blacksquare$$

⊙ **Computer Example CB.1**

Express the following numbers in polar form: **(a)** $2 + j3$ **(b)** $-2 + j1$

MATLAB function *cart2pol(a,b)* can be used to convert the complex number $a + jb$ to its polar form.

(a)

```
[Zangle_in_rad,Zmag]=cart2pol(2,3)
Zangle_in_rad = 0.9828
Zmag =3.6056
Zangle_in_deg=Zangle_in_rad*(180/pi)
Zangle_in_deg=56.31
```

Therefore

$$z = 2 + j3 = 3.6056e^{j56.31^\circ}$$

(b)

```
[Zangle_in_rad,Zmag]=cart2pol(-2,1)
Zangle_in_rad = 2.6779
Zmag =2.2361
Zangle_in_deg=Zangle_in_rad*(180/pi)
Zangle_in_deg=153.4349
```

Therefore

$$z = -2 + j1 = 2.2361e^{j153.4349^\circ}$$

Note that MATLAB automatically takes care of the quadrant in which the complex number lies. ⊙

■ **Example B.2**

Represent the following numbers in the complex plane and express them in Cartesian form: **(a)** $2e^{j\pi/3}$ **(b)** $4e^{-j3\pi/4}$ **(c)** $2e^{j\pi/2}$ **(d)** $3e^{-j3\pi}$ **(e)** $2e^{j4\pi}$ **(f)** $2e^{-j4\pi}$.

(a) $2e^{j\pi/3} = 2\left(\cos\frac{\pi}{3} + j\sin\frac{\pi}{3}\right) = 1 + j\sqrt{3}$ (see Fig. B.5a)

(b) $4e^{-j3\pi/4} = 4\left(\cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4}\right) = -2\sqrt{2} - j2\sqrt{2}$ (see Fig. B.5b)

(c) $2e^{j\pi/2} = 2\left(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}\right) = 2(0 + j1) = j2$ (see Fig. B.5c)

(d) $3e^{-j3\pi} = 3(\cos 3\pi - j\sin 3\pi) = 3(-1 + j0) = -3$ (see Fig. B.5d)

(e) $2e^{j4\pi} = 2(\cos 4\pi + j\sin 4\pi) = 2(1 + j0) = 2$ (see Fig. B.5e)

(f) $2e^{-j4\pi} = 2(\cos 4\pi - j\sin 4\pi) = 2(1 - j0) = 2$ (see Fig. B.5f) ■

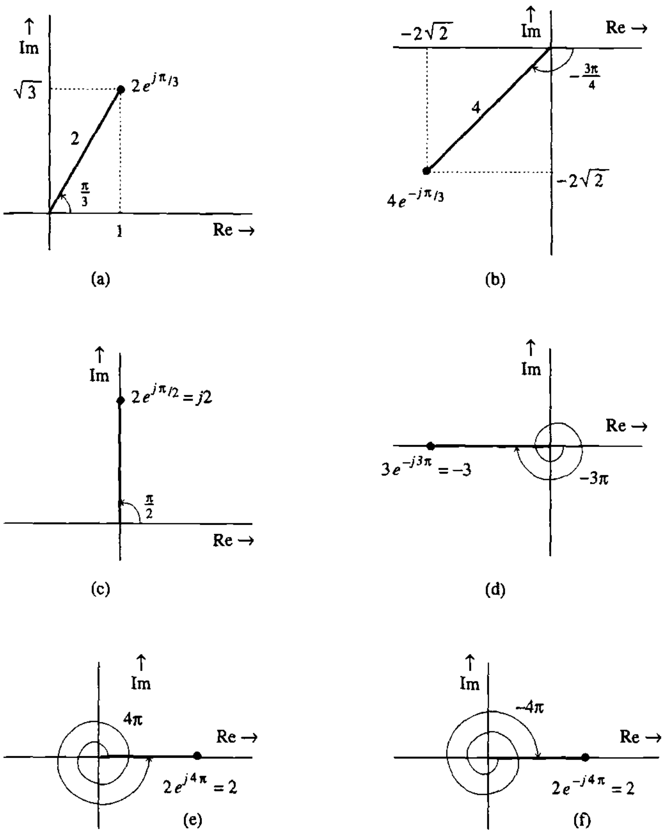

Fig. B.5 From polar to Cartesian form.

⊙ **Computer Example CB.2**

Represent $4e^{-j\frac{3\pi}{4}}$ in Cartesian form.
MATLAB function pol2cart(θ, r) converts the complex number $re^{j\theta}$ to Cartesian form.

```
[Zreal,Zimag]=pol2cart(-3*pi/4,4)
Zreal=-2.8284
Zimag=-2.8284
```

Therefore

$$4e^{-j\frac{3\pi}{4}} = -2.8284 - j2.8284$$ ⊙

Arithmetical Operations, Powers, and Roots of Complex Numbers

To perform addition and subtraction, complex numbers should be expressed in Cartesian form. Thus, if

$$z_1 = 3 + j4 = 5e^{j53.1^\circ}$$

and

$$z_2 = 2 + j3 = \sqrt{13}e^{j56.3^\circ}$$

then

$$z_1 + z_2 = (3 + j4) + (2 + j3) = 5 + j7$$

If z_1 and z_2 are given in polar form, we would need to convert them into Cartesian form for the purpose of adding (or subtracting). Multiplication and division, however, can be carried out in either Cartesian or polar form, although the latter proves to be much more convenient. This is because if z_1 and z_2 are expressed in polar form as

$$z_1 = r_1 e^{j\theta_1} \qquad \text{and} \qquad z_2 = r_2 e^{j\theta_2}$$

then

$$z_1 z_2 = \left(r_1 e^{j\theta_1}\right)\left(r_2 e^{j\theta_2}\right) = r_1 r_2 e^{j(\theta_1+\theta_2)} \tag{B.15a}$$

and

$$\frac{z_1}{z_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1-\theta_2)} \tag{B.15b}$$

Moreover,

$$z^n = \left(re^{j\theta}\right)^n = r^n e^{jn\theta} \tag{B.15c}$$

and

$$z^{1/n} = \left(re^{j\theta}\right)^{1/n} = r^{1/n} e^{j\theta/n} \tag{B.15d}$$

This shows that the operations of multiplication, division, powers, and roots can be carried out with remarkable ease when the numbers are in polar form.

■ **Example B.3**

Determine $z_1 z_2$ and z_1/z_2 for the numbers

$$z_1 = 3 + j4 = 5e^{j53.1^\circ}$$

$$z_2 = 2 + j3 = \sqrt{13}e^{j56.3^\circ}$$

We shall solve this problem in both polar and Cartesian forms.

Multiplication: Cartesian Form

$$z_1 z_2 = (3 + j4)(2 + j3) = (6 - 12) + j(8 + 9) = -6 + j17$$

Multiplication: Polar Form

$$z_1 z_2 = \left(5e^{j53.1^\circ}\right)\left(\sqrt{13}e^{j56.3^\circ}\right) = 5\sqrt{13}e^{j109.4^\circ}$$

Division: Cartesian Form

$$\frac{z_1}{z_2} = \frac{3 + j4}{2 + j3}$$

In order to eliminate the complex number in the denominator, we multiply both the numerator and the denominator of the right-hand side by $2 - j3$, the denominator's conjugate. This yields

$$\frac{z_1}{z_2} = \frac{(3+j4)(2-j3)}{(2+j3)(2-j3)} = \frac{18-j1}{2^2+3^2} = \frac{18-j1}{13} = \frac{18}{13} - j\frac{1}{13}$$

Division: Polar Form

$$\frac{z_1}{z_2} = \frac{5e^{j53.1^\circ}}{\sqrt{13}e^{j56.3^\circ}} = \frac{5}{\sqrt{13}}e^{j(53.1^\circ - 56.3^\circ)} = \frac{5}{\sqrt{13}}e^{-j3.2^\circ} \quad \blacksquare$$

It is clear from this example that multiplication and division are easier to accomplish in polar form than in Cartesian form.

■ **Example B.4**

For $z_1 = 2e^{j\pi/4}$ and $z_2 = 8e^{j\pi/3}$, find **(a)** $2z_1 - z_2$ **(b)** $\frac{1}{z_1}$ **(c)** $\frac{z_1}{z_2^2}$ **(d)** $\sqrt[3]{z_2}$

(a) Since subtraction cannot be performed directly in polar form, we convert z_1 and z_2 to Cartesian form:

$$z_1 = 2e^{j\pi/4} = 2\left(\cos\tfrac{\pi}{4} + j\sin\tfrac{\pi}{4}\right) = \sqrt{2} + j\sqrt{2}$$

$$z_2 = 8e^{j\pi/3} = 8\left(\cos\tfrac{\pi}{3} + j\sin\tfrac{\pi}{3}\right) = 4 + j4\sqrt{3}$$

Therefore,

$$\begin{aligned} 2z_1 - z_2 &= 2(\sqrt{2} + j\sqrt{2}) - (4 + j4\sqrt{3}) \\ &= (2\sqrt{2} - 4) + j(2\sqrt{2} - 4\sqrt{3}) \\ &= -1.17 - j4.1 \end{aligned}$$

(b)

$$\frac{1}{z_1} = \frac{1}{2e^{j\pi/4}} = \frac{1}{2}e^{-j\pi/4}$$

(c)

$$\frac{z_1}{z_2^2} = \frac{2e^{j\pi/4}}{(8e^{j\pi/3})^2} = \frac{2e^{j\pi/4}}{64e^{j2\pi/3}} = \frac{1}{32}e^{j(\frac{\pi}{4} - \frac{2\pi}{3})} = \frac{1}{32}e^{-j\frac{5\pi}{12}}$$

(d)

$$\sqrt[3]{z_2} = z_2^{1/3} = \left(8e^{j\pi/3}\right)^{\frac{1}{3}} = 8^{\frac{1}{3}}\left(e^{j\pi/3}\right)^{1/3} = 2e^{j\pi/9} \quad \blacksquare$$

⊙ **Computer Example CB.3**

Determine $z_1 z_2$ and z_1/z_2 if $z_1 = 3 + j4$ and $z_2 = 2 + j3$

Multiplication and division: Cartesian Form

```
z1=3+j*4; z2=2+j*3;
z1z2=z1*z2
z1z2=-6.000+17.0000i
z1_over_z2=z1/z2
z1_over_z2=1.3486-0.0769i
```

Therefore

$$(3+j4)(2+j3) = -6 + j17 \quad \text{and} \quad (3+j4)/(2+j3) = 1.3486 - 0.0769 \quad \odot$$

Example B.5

Consider $F(\omega)$, a complex function of a real variable ω:

$$F(\omega) = \frac{2 + j\omega}{3 + j4\omega} \tag{B.16a}$$

(a) Express $F(\omega)$ in Cartesian form, and find its real and imaginary parts. **(b)** Express $F(\omega)$ in polar form, and find its magnitude $|F(\omega)|$ and angle $\angle F(\omega)$.

(a) To obtain the real and imaginary parts of $F(\omega)$, we must eliminate imaginary terms in the denominator of $F(\omega)$. This is readily done by multiplying both the numerator and denominator of $F(\omega)$ by $3 - j4\omega$, the conjugate of the denominator $3 + j4\omega$ so that

$$F(\omega) = \frac{(2 + j\omega)(3 - j4\omega)}{(3 + j4\omega)(3 - j4\omega)} = \frac{(6 + 4\omega^2) - j5\omega}{9 + 16\omega^2} = \frac{6 + 4\omega^2}{9 + 16\omega^2} - j\frac{5\omega}{9 + \omega^2} \tag{B.16b}$$

This is the Cartesian form of $F(\omega)$. Clearly the real and imaginary parts $F_r(\omega)$ and $F_i(\omega)$ are given by

$$F_r(\omega) = \frac{6 + 4\omega^2}{9 + 16\omega^2}, \qquad F_i(\omega) = \frac{-5\omega}{9 + 16\omega^2}$$

(b)

$$\begin{aligned} F(\omega) = \frac{2 + j\omega}{3 + j4\omega} &= \frac{\sqrt{4 + \omega^2}\, e^{j\tan^{-1}\left(\frac{\omega}{2}\right)}}{\sqrt{9 + 16\omega^2}\, e^{j\tan^{-1}\left(\frac{4\omega}{3}\right)}} \\ &= \sqrt{\frac{4 + \omega^2}{9 + 16\omega^2}}\, e^{j\left[\tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{4\omega}{3}\right)\right]} \end{aligned} \tag{B.16c}$$

This is the polar representation of $F(\omega)$. Observe that

$$|F(\omega)| = \sqrt{\frac{4 + \omega^2}{9 + 16\omega^2}}, \qquad \angle F(\omega) = \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{4\omega}{3}\right) \tag{B.17}$$

■

B.2 Sinusoids

Consider the sinusoid

$$f(t) = C\cos\left(2\pi\mathcal{F}_0 t + \theta\right) \tag{B.18}$$

We know that

$$\cos\varphi = \cos\left(\varphi + 2n\pi\right) \qquad n = 0, \pm 1, \pm 2, \pm 3, \cdots$$

Therefore, $\cos\varphi$ repeats itself for every change of 2π in the angle φ. For the sinusoid in Eq. (B.18), the angle $2\pi\mathcal{F}_0 t + \theta$ changes by 2π when t changes by $1/\mathcal{F}_0$. Clearly, this sinusoid repeats every $1/\mathcal{F}_0$ seconds. As a result, there are $\mathcal{F}_0$ repetitions per second. This is the **frequency** of the sinusoid, and the repetition interval T_0 given by

$$T_0 = \frac{1}{\mathcal{F}_0} \tag{B.19}$$

is the **period**. For the sinusoid in Eq. (B.18), C is the **amplitude**, $\mathcal{F}_0$ is the **frequency** (in **Hertz**), and θ is the phase. Let us consider two special cases of this sinusoid when $\theta = 0$ and $\theta = -\pi/2$ as follows:

(a) $f(t) = C \cos 2\pi \mathcal{F}_0 t \qquad (\theta = 0)$

(b) $f(t) = C \cos \left(2\pi \mathcal{F}_0 t - \frac{\pi}{2}\right) = C \sin 2\pi \mathcal{F}_0 t \qquad (\theta = -\pi/2)$

The angle or phase can be expressed in units of degrees or radians. Although the radian is the proper unit, in this book we shall often use the degree unit because students generally have a better feel for the relative magnitudes of angles when expressed in degrees rather than in radians. For example, we relate better to the angle 24° than to 0.419 radians. Remember, however, when in doubt, use the radian unit and, above all, be consistent. In other words, in a given problem or an expression do not mix the two units.

It is convenient to use the variable ω_0 (*radian frequency*) to express $2\pi\mathcal{F}_0$:

$$\omega_0 = 2\pi \mathcal{F}_0 \tag{B.20}$$

With this notation, the sinusoid in Eq. (B.18) can be expressed as

$$f(t) = C \cos (\omega_0 t + \theta)$$

in which the period T_0 is given by [see Eqs. (B.19) and (B.20)]

$$T_0 = \frac{1}{\omega_0/2\pi} = \frac{2\pi}{\omega_0} \tag{B.21a}$$

and

$$\omega_0 = \frac{2\pi}{T_0} \tag{B.21b}$$

In future discussions, we shall often refer to ω_0 as the frequency of the signal $\cos(\omega_0 t + \theta)$, but it should be clearly understood that the frequency of this sinusoid is $\mathcal{F}_0$ Hz ($\mathcal{F}_0 = \omega_0/2\pi$), and ω_0 is actually the **radian frequency**.

The signals $C \cos \omega_0 t$ and $C \sin \omega_0 t$ are illustrated in Figs. B.6a and B.6b respectively. A general sinusoid $C \cos(\omega_0 t + \theta)$ can be readily sketched by shifting the signal $C \cos \omega_0 t$ in Fig. B.6a by the appropriate amount. Consider, for example,

$$f(t) = C \cos (\omega_0 t - 60^\circ)$$

This signal can be obtained by shifting (delaying) the signal $C \cos \omega_0 t$ (Fig. B.6a) to the right by a phase (angle) of 60°. We know that a sinusoid undergoes a 360° change of phase (or angle) in one cycle. A quarter-cycle segment corresponds to a 90° change of angle. Therefore, an angle of 60° corresponds to two-thirds of a quarter-cycle segment. We therefore shift (delay) the signal in Fig. B.6a by two-thirds of a quarter-cycle segment to obtain $C \cos(\omega_0 t - 60^\circ)$, as shown in Fig. B.6c.

Observe that if we delay $C \cos \omega_0 t$ in Fig. B.6a by a quarter-cycle (angle of 90° or $\pi/2$ radians), we obtain the signal $C \sin \omega_0 t$, depicted in Fig. B.6b. This verifies the well-known trigonometric identity

$$C \cos \left(\omega_0 t - \frac{\pi}{2}\right) = C \sin \omega_0 t \tag{B.22a}$$

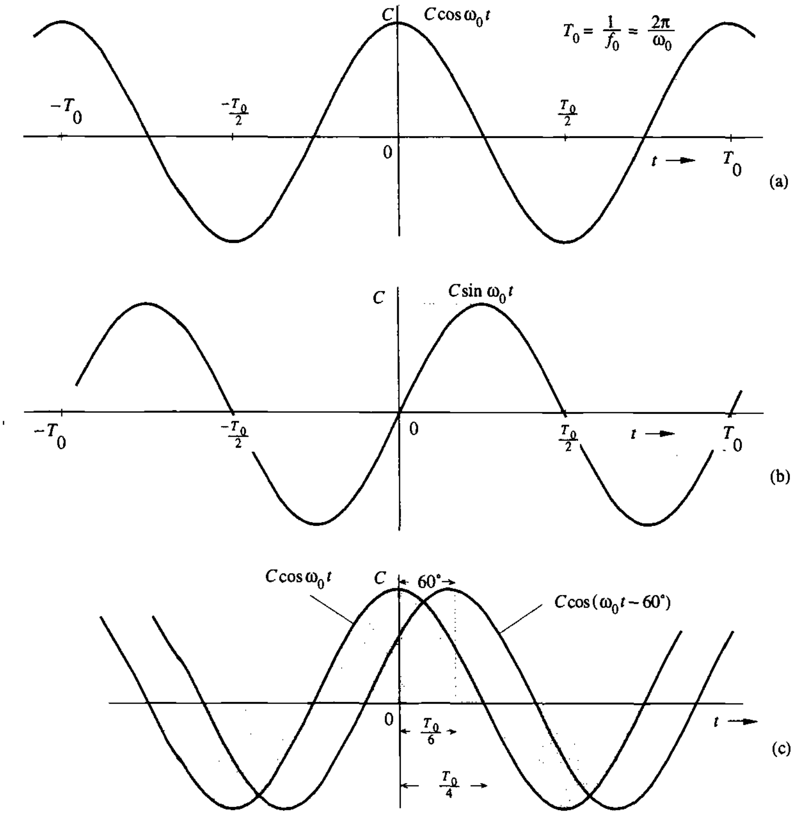

Fig. B.6 Sketching a sinusoid.

Alternatively, if we advance $C\sin\omega_0 t$ by a quarter-cycle, we obtain $C\cos\omega_0 t$. Therefore,

$$C\sin\left(\omega_0 t + \tfrac{\pi}{2}\right) = C\cos\omega_0 t \tag{B.22b}$$

This observation means $\sin\omega_0 t$ lags $\cos\omega_0 t$ by $90°(\pi/2$ radians), or $\cos\omega_0 t$ leads $\sin\omega_0 t$ by $90°$.

B.2-1 Addition of Sinusoids

Two sinusoids having the same frequency but different phases add to form a single sinusoid of the same frequency. This fact is readily seen from the well-known trigonometric identity

$$\begin{aligned} C\cos(\omega_0 t + \theta) &= C\cos\theta\cos\omega_0 t - C\sin\theta\sin\omega_0 t \\ &= a\cos\omega_0 t + b\sin\omega_0 t \end{aligned} \tag{B.23a}$$

in which

$$a = C\cos\theta, \qquad b = -C\sin\theta$$

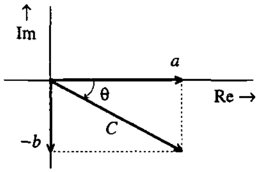

Fig. B.7 Phasor addition of sinusoids.

Therefore,

$$C = \sqrt{a^2 + b^2} \tag{B.23b}$$

$$\theta = \tan^{-1}\left(\frac{-b}{a}\right) \tag{B.23c}$$

Equations (B.23b) and (B.23c) show that C and θ are the magnitude and angle, respectively, of a complex number $a - jb$. In other words, $a - jb = Ce^{j\theta}$. Hence, to find C and θ, we convert $a - jb$ to polar form and the magnitude and the angle of the resulting polar number are C and θ, respectively.

To summarize,

$$a \cos \omega_0 t + b \sin \omega_0 t = C \cos (\omega_0 t + \theta)$$

in which C and θ are given by Eqs. (B.23b) and (B.23c), respectively. These happen to be the magnitude and angle, respectively, of $a - jb$.

The process of adding two sinusoids with the same frequency can be clarified by using **phasors** to represent sinusoids. We represent the sinusoid $C \cos (\omega_0 t + \theta)$ by a phasor of length C at an angle θ with the horizontal axis. Clearly, the sinusoid $a \cos \omega_0 t$ is represented by a horizontal phasor of length a ($\theta = 0$), while $b \sin \omega_0 t = b \cos (\omega_0 t - \frac{\pi}{2})$ is represented by a vertical phasor of length b at an angle $-\pi/2$ with the horizontal (Fig. B.7). Adding these two phasors results in a phasor of length C at an angle θ, as depicted in Fig. B.7. From this figure, we verify the values of C and θ found in Eqs. (B.23b) and (B.23c), respectively.

Proper care should be exercised in computing θ. Recall that $\tan^{-1}(\frac{-b}{a}) \neq \tan^{-1}(\frac{b}{-a})$. Similarly, $\tan^{-1}(\frac{-b}{-a}) \neq \tan^{-1}(\frac{b}{a})$. Electronic calculators cannot make this distinction. When calculating such an angle, it is advisable to note the quadrant where the angle lies and not to rely exclusively on an electronic calculator. A foolproof method is to convert the complex number $a - jb$ to polar form. The magnitude of the resulting polar number is C and the angle is θ. The following examples clarify this point.

■ **Example B.6**

In the following cases, express $f(t)$ as a single sinusoid:

(a) $f(t) = \cos \omega_0 t - \sqrt{3} \sin \omega_0 t$

(b) $f(t) = -3 \cos \omega_0 t + 4 \sin \omega_0 t$

(a) In this case, $a = 1, b = -\sqrt{3}$, and from Eqs. (B.23)

$$C = \sqrt{1^2 + (\sqrt{3})^2} = 2$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = 60^\circ$$

Therefore,

$$f(t) = 2\cos(\omega_0 t + 60^\circ)$$

We can verify this result by drawing phasors corresponding to the two sinusoids. The sinusoid $\cos \omega_0 t$ is represented by a phasor of unit length at a zero angle with the horizontal. The phasor $\sin \omega_0 t$ is represented by a unit phasor at an angle of -90° with the horizontal. Therefore, $-\sqrt{3}\sin \omega_0 t$ is represented by a phasor of length $\sqrt{3}$ at 90° with the horizontal, as depicted in Fig. B.8a. The two phasors added yield a phasor of length 2 at 60° with the horizontal (also shown in Fig. B.8a). Therefore,

$$f(t) = 2\cos(\omega_0 t + 60^\circ)$$

Alternately, we note that $a - jb = 1 + j\sqrt{3} = 2e^{j\pi/3}$. Hence, $C = 2$ and $\theta = \pi/3$.

Observe that a phase shift of $\pm\pi$ amounts to multiplication by -1. Therefore, $f(t)$ can also be expressed alternatively as

$$\begin{aligned} f(t) &= -2\cos(\omega_0 t + 60^\circ \pm 180^\circ) \\ &= -2\cos(\omega_0 t - 120^\circ) \\ &= -2\cos(\omega_0 t + 240^\circ) \end{aligned}$$

In practice, an expression with an angle whose numerical value is less than 180° is preferred.

(b) In this case, $a = -3, b = 4$, and from Eqs. (B.23)

$$\begin{aligned} C &= \sqrt{(-3)^2 + 4^2} = 5 \\ \theta &= \tan^{-1}\left(\tfrac{-4}{-3}\right) = -126.9^\circ \end{aligned}$$

Observe that

$$\tan^{-1}\left(\tfrac{-4}{-3}\right) \neq \tan^{-1}\left(\tfrac{4}{3}\right) = 53.1^\circ$$

Therefore,

$$f(t) = 5\cos(\omega_0 t - 126.9^\circ)$$

This result is readily verified in the phasor diagram in Fig. B.8b. Alternately, $a - jb = -3 - j4 = 5e^{-j126.9^\circ}$. Hence, $C = 5$ and $\theta = -126.9^\circ$. ■

⊙ Computer Example CB.4

Express $f(t) = -3\cos \omega_0 t + 4\sin \omega_0 t$ as a single sinusoid.

Recall that $a\cos \omega_0 t + b\sin \omega_0 t = C\cos[\omega_0 t + \tan^{-1}(-b/a)]$. Hence, the amplitude C and the angle θ of the resulting sinusoid are the magnitude and angle of a complex number $a - jb$. We use the 'cart2pol' function to convert it to the polar form to obtain C and θ.

```
a=-3;b=4;
[theta,C]=cart2pol(a,-b);
Theta_deg=(180/pi)*theta;
C,Theta_deg
C=5
Theta_deg=-126.8699
```

Therefore

$$-3\cos \omega_0 t + 4\sin \omega_0 t = 5\cos(\omega_0 t - 126.8699^\circ)$$ ⊙

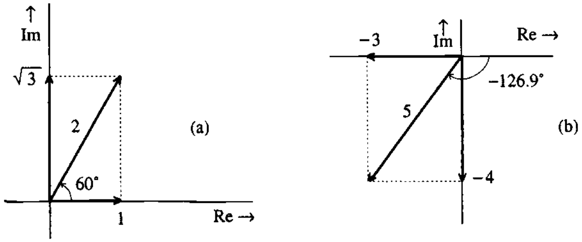

Fig. B.8 Phasor addition of sinusoids in Example B.6.

We can also perform the reverse operation, expressing

$$f(t) = C \cos(\omega_0 t + \theta)$$

in terms of $\cos \omega_0 t$ and $\sin \omega_0 t$ using the trigonometric identity

$$C \cos(\omega_0 t + \theta) = C \cos\theta \cos\omega_0 t - C \sin\theta \sin\omega_0 t$$

For example,

$$10 \cos(\omega_0 t - 60°) = 5 \cos\omega_0 t + 5\sqrt{3} \sin\omega_0 t$$

Sinusoids in Terms of Exponentials: Euler's Formula

Sinusoids can be expressed in terms of exponentials using Euler's formula [see Eq. (B.3)]

$$\cos\varphi = \frac{1}{2}\left(e^{j\varphi} + e^{-j\varphi}\right) \tag{B.24a}$$

$$\sin\varphi = \frac{1}{2j}\left(e^{j\varphi} - e^{-j\varphi}\right) \tag{B.24b}$$

Inversion of these equations yields

$$e^{j\varphi} = \cos\varphi + j\sin\varphi \tag{B.25a}$$

$$e^{-j\varphi} = \cos\varphi - j\sin\varphi \tag{B.25b}$$

B.3 Sketching Signals

In this section we discuss the sketching of a few useful signals, starting with exponentials.

B.3-1 Monotonic Exponentials

The signal e^{-at} decays monotonically, and the signal e^{at} grows monotonically with t (assuming $a > 0$) as depicted in Fig. B.9. For the sake of simplicity, we shall consider an exponential e^{-at} starting at $t = 0$, as shown in Fig. B.10a.

The signal e^{-at} has a unit value at $t = 0$. At $t = 1/a$, the value drops to $1/e$ (about 37% of its initial value), as illustrated in Fig. B.10a. This time interval over which the exponential reduces by a factor e (that is, drops to about 37% of its value) is known as the **time constant** of the exponential. Therefore, the time

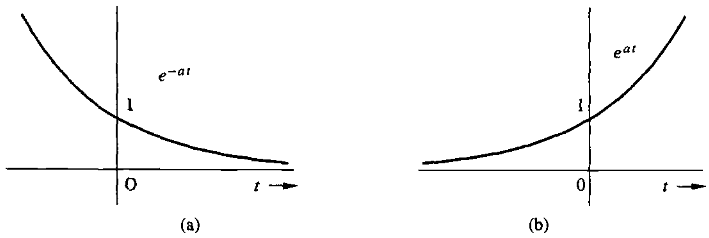

Fig. B.9 Monotonic exponentials.

constant of e^{-at} is $1/a$. Observe that the exponential is reduced to 37% of its initial value over any time interval of duration $1/a$. This can be shown by considering any set of instants t_1 and t_2 separated by one time constant so that

$$t_2 - t_1 = \frac{1}{a}$$

Now the ratio of e^{-at_2} to e^{-at_1} is given by

$$\frac{e^{-at_2}}{e^{-at_1}} = e^{-a(t_2-t_1)} = \frac{1}{e} \approx 0.37$$

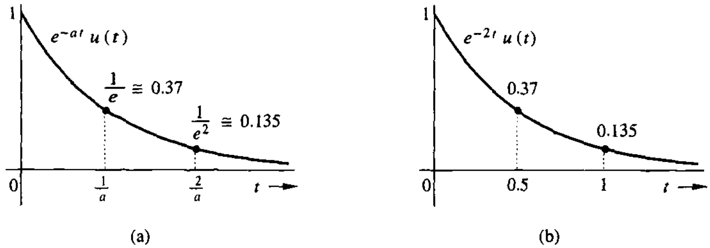

Fig. B.10 (a) Sketching e^{-at} (b) sketching e^{-2t}.

We can use this fact to sketch an exponential quickly. For example, consider

$$f(t) = e^{-2t}$$

The time constant in this case is $1/2$. The value of $f(t)$ at $t = 0$ is 1. At $t = 1/2$ (one time constant) it is $1/e$ (about 0.37). The value of $f(t)$ continues to drop further by the factor $1/e$ (37%) over the next half-second interval (one time constant). Thus $f(t)$ at $t = 1$ is $(1/e)^2$. Continuing in this manner, we see that $f(t) = (1/e)^3$ at $t = 3/2$ and so on. A knowledge of the values of $f(t)$ at $t = 0$, 0.5, 1, and 1.5 allows us to sketch the desired signal† as shown in Fig. B.10b. For a monotonically

†If we wish to refine the sketch further, we could consider intervals of half the time constant over which the signal decays by a factor $1/\sqrt{e}$. Thus, at $t = 0.25$, $f(t) = 1/\sqrt{e}$, and at $t = 0.75$, $f(t) = 1/e\sqrt{e}$, etc.

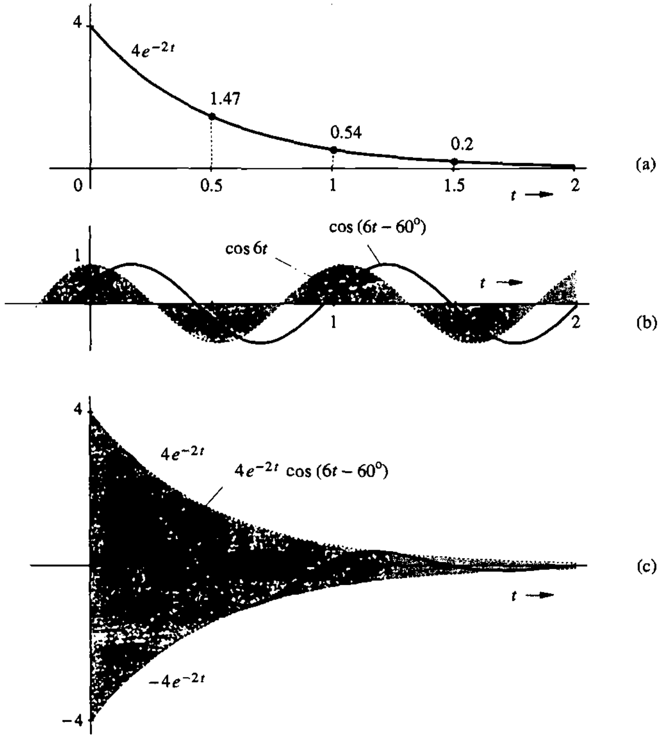

Fig. B.11 Sketching an exponentially varying sinusoid

growing exponential e^{at}, the waveform increases by a factor e over each interval of $1/a$ seconds.

B.3-2 The Exponentially Varying Sinusoid

We now discuss sketching an exponentially varying sinusoid

$$f(t) = Ae^{-at} \cos\,(\omega_0 t + \theta)$$

Let us consider a specific example

$$f(t) = 4e^{-2t} \cos\,(6t - 60^\circ) \tag{B.26}$$

We shall sketch $4e^{-2t}$ and $\cos\,(6t - 60^\circ)$ separately and then multiply them.

(i) Sketching $4e^{-2t}$

This monotonically decaying exponential has a time constant of 1/2 second and an initial value of 4 at $t = 0$. Therefore, its values at $t = 0.5$, 1, 1.5, and 2 are $4/e$, $4/e^2$, $4/e^3$, and $4/e^4$, or about 1.47, 0.54, 0.2, and 0.07 respectively. Using these values as a guide, we sketch $4e^{-2t}$, as illustrated in Fig. B.11a.

(ii) Sketching $\cos(6t - 60°)$

The procedure for sketching $\cos(6t - 60°)$ is discussed in Sec. B.2 (Fig. B.6c). Here the period of the sinusoid is $T_0 = 2\pi/6 \approx 1$, and there is a phase delay of 60°, or two-thirds of a quarter-cycle, which is equivalent to about a $(60/360)(1) \approx 1/6$ second delay (see Fig. B.11b).

(iii) Sketching $4e^{-2t}\cos(6t - 60°)$

We now multiply the waveforms in (i) and (ii). The multiplication amounts to forcing the amplitude of the sinusoid $\cos(6t - 60°)$ to decrease exponentially with a time constant of 0.5. The initial amplitude (at $t = 0$) is 4, decreasing to $4/e$ (=1.47) at $t = 0.5$, to $1.47/e$ (=0.54) at $t = 1$, and so on. This is depicted in Fig. B.11c. Note that at the instants where $\cos(6t - 60°)$ has a value of unity (peak amplitude),

$$4e^{-2t}\cos(6t - 60°) = 4e^{-2t} \tag{B.27}$$

Therefore, $4e^{-2t}\cos(6t - 60°)$ touches $4e^{-2t}$ at those instants where the sinusoid $\cos(6t - 60°)$ is at its positive peaks. Clearly $4e^{-2t}$ is an envelope for positive amplitudes of $4e^{-2t}\cos(6t - 60°)$. Similarly, at those instants where the sinusoid $\cos(6t - 60°)$ has a value of -1 (negative peak amplitude),

$$4e^{-2t}\cos(6t - 60°) = -4e^{-2t} \tag{B.28}$$

and $4e^{-2t}\cos(6t - 60°)$ touches $-4e^{-2t}$ at its negative peaks. Therefore, $-4e^{-2t}$ is an envelope for negative amplitudes of $4e^{-2t}\cos(6t - 60°)$. Thus, to sketch $4e^{-2t}\cos(6t - 60°)$, we first draw the envelopes $4e^{-2t}$ and $-4e^{-2t}$ (the mirror image of $4e^{-2t}$ about the horizontal axis), and then sketch the sinusoid $\cos(6t - 60°)$, with these envelopes acting as constraints on the sinusoid's amplitude (see Fig. B.11c).

In general, $Ke^{-at}\cos(\omega_0 t + \theta)$ can be sketched in this manner, with Ke^{-at} and $-Ke^{-at}$ constraining the amplitude of $\cos(\omega_0 t + \theta)$.

B.4 Cramer's Rule

This is a very convenient rule used to solve simultaneous linear equations. Consider a set of n linear simultaneous equations in n unknowns $x_1, x_2, \ldots, x_n$:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= y_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= y_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= y_n
\end{aligned} \tag{B.29}$$

These equations can be expressed in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \tag{B.30}$$

We denote the matrix on the left-hand side formed by the elements a_{ij} as $\mathbf{A}$. The determinant of $\mathbf{A}$ is denoted by $|\mathbf{A}|$. If the determinant $|\mathbf{A}|$ is not zero, the set of equations (B.29) has a unique solution given by Cramer's formula

$$x_k = \frac{|\mathbf{D}_k|}{|\mathbf{A}|} \qquad k = 1, 2, \ldots, n \tag{B.31}$$

where $|\mathbf{D}_k|$ is obtained by replacing the kth column of $|\mathbf{A}|$ by the column on the right-hand side of Eq. (B.30) (with elements $y_1, y_2, \ldots, y_n$).

We shall demonstrate the use of this rule with an example.

Example B.7

Using Cramer's rule, solve the following simultaneous linear equations in three unknowns:

$$2x_1 + x_2 + x_3 = 3$$

$$x_1 + 3x_2 - x_3 = 7$$

$$x_1 + x_2 + x_3 = 1$$

In matrix form these equations can be expressed as

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}$$

Here,

$$|\mathbf{A}| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 4$$

Since $|\mathbf{A}| = 4 \neq 0$, a unique solution exists for x_1, x_2, and x_3. This solution is provided by Cramer's rule (B.31) as follows:

$$x_1 = \frac{1}{|\mathbf{A}|} \begin{vmatrix} 3 & 1 & 1 \\ 7 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \frac{8}{4} = 2$$

$$x_2 = \frac{1}{|\mathbf{A}|} \begin{vmatrix} 2 & 3 & 1 \\ 1 & 7 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \frac{4}{4} = 1$$

$$x_3 = \frac{1}{|\mathbf{A}|}\begin{vmatrix} 2 & 1 & 3 \\ 1 & 3 & 7 \\ 1 & 1 & 1 \end{vmatrix} = \frac{-8}{4} = -2 \qquad \blacksquare$$

⊙ **Example CB.5**

Using a Computer, solve Example B.7.

```
A = [2 1 1;1 3 -1;1 1 1]; b=[3 7 1]';
for k=1:3
A1=A;
A1(:,k)=b;
D=A1;
x(k)=det(D)/det(A);
end
x=x'
x = 2
     1
    -2
```
⊙

B.5 Partial Fraction Expansion

In the analysis of linear time-invariant systems, we encounter functions that are ratios of two polynomials in a certain variable, say x. Such functions are known as **rational functions**. A rational function $F(x)$ can be expressed as

$$F(x) = \frac{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}{x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0} \tag{B.32}$$

$$= \frac{P(x)}{Q(x)} \tag{B.33}$$

The function $F(x)$ is **improper** if $m \geq n$ and **proper** if $m < n$. An improper function can always be separated into the sum of a polynomial in x and a proper function. Consider, for example, the function

$$F(x) = \frac{2x^3 + 9x^2 + 11x + 2}{x^2 + 4x + 3} \tag{B.34a}$$

Because this is an improper function, we divide the numerator by the denominator until the remainder has a lower degree than the denominator.

$$\begin{array}{r l} & 2x + 1 \\ x^2 + 4x + 3\,) & \overline{2x^3 + 9x^2 + 11x + 2} \\ & \underline{2x^3 + 8x^2 + 6x} \\ & \qquad\quad x^2 + 5x + 2 \\ & \qquad\quad \underline{x^2 + 4x + 3} \\ & \qquad\qquad\quad x - 1 \end{array}$$

Therefore, $F(x)$ can be expressed as

$$F(x) = \frac{2x^3 + 9x^2 + 11x + 2}{x^2 + 4x + 3} = \underbrace{2x + 1}_{\text{polynomial in } x} + \underbrace{\frac{x - 1}{x^2 + 4x + 3}}_{\text{proper function}} \tag{B.34b}$$

A proper function can be further expanded into partial fractions. The remaining discussion in this section is concerned with various ways of doing this.

B.5-1 Partial Fraction Expansion: Method of Clearing Fractions

This method consists of writing a rational function as a sum of appropriate partial fractions with unknown coefficients, which are determined by clearing fractions and equating the coefficients of similar powers on the two sides. This procedure is demonstrated by the following example.

■ **Example B.8**

Expand the following rational function $F(x)$ into partial fractions:

$$F(x) = \frac{x^3 + 3x^2 + 4x + 6}{(x+1)(x+2)(x+3)^2}$$

This function can be expressed as a sum of partial fractions with denominators $(x + 1)$, $(x + 2)$, $(x + 3)$, and $(x + 3)^2$, as shown below.

$$F(x) = \frac{x^3 + 3x^2 + 4x + 6}{(x+1)(x+2)(x+3)^2} = \frac{k_1}{x+1} + \frac{k_2}{x+2} + \frac{k_3}{x+3} + \frac{k_4}{(x+3)^2}$$

To determine the unknowns k_1, k_2, k_3, and k_4 we clear fractions by multiplying both sides by $(x + 1)(x + 2)(x + 3)^2$ to obtain

$$\begin{aligned} x^3 + 3x^2 + 4x + 6 &= k_1(x^3 + 8x^2 + 21x + 18) + k_2(x^3 + 7x^2 + 15x + 9) \\ &\quad + k_3(x^3 + 6x^2 + 11x + 6) + k_4(x^2 + 3x + 2) \\ &= x^3(k_1 + k_2 + k_3) + x^2(8k_1 + 7k_2 + 6k_3 + k_4) \\ &\quad + x(21k_1 + 15k_2 + 11k_3 + 3k_4) + (18k_1 + 9k_2 + 6k_3 + 2k_4) \end{aligned}$$

Equating coefficients of similar powers on both sides yields

$$\begin{aligned} k_1 + k_2 + k_3 &= 1 \\ 8k_1 + 7k_2 + 6k_3 + k_4 &= 3 \\ 21k_1 + 15k_2 + 11k_3 + 3k_4 &= 4 \\ 18k_1 + 9k_2 + 6k_3 + 2k_4 &= 6 \end{aligned}$$

Solution of these four simultaneous equations yields

$$k_1 = 1, \qquad k_2 = -2, \qquad k_3 = 2, \qquad k_4 = -3$$

Therefore,

$$F(x) = \frac{1}{x+1} - \frac{2}{x+2} + \frac{2}{x+3} - \frac{3}{(x+3)^2} \qquad ■$$

Although this method is straightforward and applicable to all situations, it is not necessarily the most efficient. We now discuss other methods which can reduce numerical work considerably.

B.5-2 Partial Fractions: The Heaviside "Cover-Up" Method

1. Unrepeated Factors of $Q(x)$

We shall first consider the partial fraction expansion of $F(x) = P(x)/Q(x)$, in which all the factors of $Q(x)$ are unrepeated. Consider the proper function

$$F(x) = \frac{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}{x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0} \qquad m < n$$

$$= \frac{P(x)}{(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)} \tag{B.35a}$$

We can show that $F(x)$ in Eq. (B.35a) can be expressed as the sum of partial fractions

$$F(x) = \frac{k_1}{x-\lambda_1} + \frac{k_2}{x-\lambda_2} + \cdots + \frac{k_n}{x-\lambda_n} \tag{B.35b}$$

To determine the coefficient k_1, we multiply both sides of Eq. (B.35b) by $x - \lambda_1$ and then let $x = \lambda_1$. This yields

$$(x-\lambda_1)F(x)\big|_{x=\lambda_1} = k_1 + \frac{k_2(x-\lambda_1)}{(x-\lambda_2)} + \frac{k_3(x-\lambda_1)}{(x-\lambda_3)} + \cdots + \frac{k_n(x-\lambda_1)}{(x-\lambda_n)}\bigg|_{x=\lambda_1}$$

On the right-hand side, all the terms except k_1 vanish. Therefore,

$$k_1 = (x-\lambda_1)F(x)\big|_{x=\lambda_1} \tag{B.36}$$

Similarly, we can show that

$$k_r = (x-\lambda_r)F(x)\big|_{x=\lambda_r} \qquad r = 1, 2, \cdots, n \tag{B.37}$$

■ **Example B.9**

Expand the following rational function $F(x)$ into partial fractions:

$$F(x) = \frac{2x^2 + 9x - 11}{(x+1)(x-2)(x+3)} = \frac{k_1}{x+1} + \frac{k_2}{x-2} + \frac{k_3}{x+3}$$

To determine k_1, we let $x = -1$ in $(x+1)F(x)$. Note that $(x+1)F(x)$ is obtained from $F(x)$ by omitting the term $(x+1)$ from its denominator. Therefore, to compute k_1 corresponding to the factor $(x+1)$, we cover up the term $(x+1)$ in the denominator of $F(x)$ and then substitute $x = -1$ in the remaining expression. (Mentally conceal the term $(x+1)$ in $F(x)$ with a finger and then let $x = -1$ in the remaining expression.) The procedure is explained step by step below.

$$F(x) = \frac{2x^2 + 9x - 11}{(x+1)(x-2)(x+3)}$$

Step 1: Cover up (conceal) the factor $(x+1)$ from $F(x)$:

$$\frac{2x^2 + 9x - 11}{\text{[redacted]}(x-2)(x+3)}$$

Step2: Substitute $x = -1$ in the remaining expression to obtain k_1:

$$k_1 = \frac{2 - 9 - 11}{(-1-2)(-1+3)} = \frac{-18}{-6} = 3$$

Similarly, to compute k_2, we cover up the factor $(x - 2)$ in $F(x)$ and let $x = 2$ in the remaining function, as shown below.

$$k_2 = \left.\frac{2x^2 + 9x - 11}{(x+1)\text{[redacted]}(x+3)}\right|_{x=2} = \frac{8 + 18 - 11}{(2+1)(2+3)} = \frac{15}{15} = 1$$

and

$$k_3 = \left.\frac{2x^2 + 9x - 11}{(x+1)(x-2)\text{[redacted]}}\right|_{x=-3} = \frac{18 - 27 - 11}{(-3+1)(-3-2)} = \frac{-20}{10} = -2$$

Therefore,

$$F(x) = \frac{2x^2 + 9x - 11}{(x+1)(x-2)(x+3)} = \frac{3}{x+1} + \frac{1}{x-2} - \frac{2}{x+3} \quad \blacksquare$$

Complex Factors in $F(x)$

The procedure above works regardless of whether the factors of $Q(x)$ are real or complex. Consider, for example,

$$\begin{aligned} F(x) &= \frac{4x^2 + 2x + 18}{(x+1)(x^2 + 4x + 13)} \\ &= \frac{4x^2 + 2x + 18}{(x+1)(x+2-j3)(x+2+j3)} \\ &= \frac{k_1}{x+1} + \frac{k_2}{x+2-j3} + \frac{k_3}{x+2+j3} \end{aligned} \tag{B.38}$$

where

$$k_1 = \left[\frac{4x^2 + 2x + 18}{\text{[redacted]}(x^2 + 4x + 13)}\right]_{x=-1} = 2$$

Similarly,

$$k_2 = \left[\frac{4x^2 + 2x + 18}{(x+1)\text{[redacted]}(x+2+j3)}\right]_{x=-2+j3} = 1 + j2 = \sqrt{5}e^{j63.43^\circ}$$

$$k_3 = \left[\frac{4x^2 + 2x + 18}{(x+1)(x+2-j3)\text{[redacted]}}\right]_{x=-2-j3} = 1 - j2 = \sqrt{5}e^{-j63.43^\circ}$$

Therefore,

$$F(x) = \frac{2}{x+1} + \frac{\sqrt{5}e^{j63.43^\circ}}{x+2-j3} + \frac{\sqrt{5}e^{-j63.43^\circ}}{x+2+j3} \tag{B.39}$$

The coefficients k_2 and k_3 corresponding to the complex conjugate factors are also conjugates of each other. This is generally true when the coefficients of a rational function are real. In such a case, we need to compute only one of the coefficients.

2. Quadratic Factors

Often we are required to combine the two terms arising from complex conjugate factors into one quadratic factor. For example, $F(x)$ in Eq. (B.38) can be expressed as

$$F(x) = \frac{4x^2+2x+18}{(x+1)(x^2+4x+13)} = \frac{k_1}{x+1} + \frac{c_1x+c_2}{x^2+4x+13}$$

The coefficient k_1 is found by the Heaviside method to be 2. Therefore,

$$\frac{4x^2+2x+18}{(x+1)(x^2+4x+13)} = \frac{2}{x+1} + \frac{c_1x+c_2}{x^2+4x+13} \tag{B.40}$$

The values of c_1 and c_2 are determined by clearing fractions and equating the coefficients of similar powers of x on both sides of the resulting equation. Clearing fractions on both sides of Eq. (B.40) yields

$$\begin{aligned} 4x^2+2x+18 &= 2(x^2+4x+13)+(c_1x+c_2)(x+1) \\ &= (2+c_1)x^2+(8+c_1+c_2)x+(26+c_2) \end{aligned} \tag{B.41}$$

Equating terms of similar powers yields $c_1 = 2$, $c_2 = -8$, and

$$\frac{4x^2+2x+18}{(x+1)(x^2+4x+13)} = \frac{2}{x+1} + \frac{2x-8}{x^2+4x+13} \tag{B.42}$$

Short-Cuts

The values of c_1 and c_2 in Eq. (B.40) can also be determined by using short-cuts. After computing $k_1 = 2$ by the Heaviside method as before, we let $x = 0$ on both sides of Eq. (B.40) to eliminate c_1. This gives us

$$\frac{18}{13} = 2 + \frac{c_2}{13}$$

Therefore,

$$c_2 = -8$$

To determine c_1, we multiply both sides of Eq. (B.40) by x and then let $x \to \infty$. Remember that when $x \to \infty$, only the terms of the highest power are significant. Therefore,

$$4 = k_1 + c_1 = 2 + c_1$$

and

$$c_1 = 2$$

In the procedure discussed here, we let $x = 0$ to determine c_2 and then multiply both sides by x and let $x \to \infty$ to determine c_1. However, nothing is sacred about these values ($x = 0$ or $x = \infty$). We use them because they reduce the number of computations involved. We could just as well use other convenient values for x, such as $x = 1$. Consider the case

$$F(x) = \frac{2x^2 + 4x + 5}{x(x^2 + 2x + 5)} = \frac{k}{x} + \frac{c_1 x + c_2}{x^2 + 2x + 5}$$

We find $k = 1$ by the Heaviside method in the usual manner. As a result,

$$\frac{2x^2 + 4x + 5}{x(x^2 + 2x + 5)} = \frac{1}{x} + \frac{c_1 x + c_2}{x^2 + 2x + 5} \tag{B.43}$$

To determine c_1 and c_2, if we try letting $x = 0$ in Eq. (B.43), we obtain ∞ on both sides. So let us choose $x = 1$. This yields

$$F(1) = \frac{11}{8} = 1 + \frac{c_1 + c_2}{8}$$

or

$$c_1 + c_2 = 3$$

We can now choose some other value for x, such as $x = 2$, to obtain one more relationship to use in determining c_1 and c_2. In this case, however, a simple method is to multiply both sides of Eq. (B.43) by x and then let $x \to \infty$. This yields

$$2 = 1 + c_1$$

so that

$$c_1 = 1 \qquad \text{and} \qquad c_2 = 2$$

Therefore,

$$F(x) = \frac{1}{x} + \frac{x + 2}{x^2 + 2x + 5}$$

B.5-3 Repeated Factors in $Q(x)$

If a function $F(x)$ has a repeated factor in its denominator, it has the form

$$F(x) = \frac{P(x)}{(x - \lambda)^r (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_j)} \tag{B.44}$$

Its partial fraction expansion is given by

$$F(x) = \frac{a_0}{(x - \lambda)^r} + \frac{a_1}{(x - \lambda)^{r-1}} + \cdots + \frac{a_{r-1}}{(x - \lambda)} + \frac{k_1}{x - \alpha_1} + \frac{k_2}{x - \alpha_2} + \cdots + \frac{k_j}{x - \alpha_j} \tag{B.45}$$

The coefficients $k_1, k_2, \ldots, k_j$ corresponding to the unrepeated factors in this equation are determined by the Heaviside method, as before [Eq. (B.37)]. To find the coefficients a_0, a_1, a_2, ..., a_{r-1}, we multiply both sides of Eq. (B.45) by $(x-\lambda)^r$. This gives us

$$(x-\lambda)^r F(x) = a_0 + a_1(x-\lambda) + a_2(x-\lambda)^2 + \cdots + a_{r-1}(x-\lambda)^{r-1} + k_1\frac{(x-\lambda)^r}{x-\alpha_1} + k_2\frac{(x-\lambda)^r}{x-\alpha_2} + \cdots + k_n\frac{(x-\lambda)^r}{x-\alpha_n} \tag{B.46}$$

If we let $x = \lambda$ on both sides of Eq. (B.46), we obtain

$$(x-\lambda)^r F(x)\big|_{x=\lambda} = a_0 \tag{B.47a}$$

Therefore, a_0 is obtained by concealing the factor $(x-\lambda)^r$ in $F(x)$ and letting $x = \lambda$ in the remaining expression (the Heaviside "cover up" method). If we take the derivative (with respect to x) of both sides of Eq. (B.46), the right-hand side is a_1+ terms containing a factor $(x-\lambda)$ in their numerators. Letting $x = \lambda$ on both sides of this equation, we obtain

$$\frac{d}{dx}\left[(x-\lambda)^r F(x)\right]\bigg|_{x=\lambda} = a_1$$

Thus, a_1 is obtained by concealing the factor $(x-\lambda)^r$ in $F(x)$, taking the derivative of the remaining expression, and then letting $x = \lambda$. Continuing in this manner, we find

$$a_j = \frac{1}{j!}\frac{d^j}{dx^j}\left[(x-\lambda)^r F(x)\right]\bigg|_{x=\lambda} \tag{B.47b}$$

Observe that $(x-\lambda)^r F(x)$ is obtained from $F(x)$ by omitting the factor $(x-\lambda)^r$ from its denominator. Therefore, the coefficient a_j is obtained by concealing the factor $(x-\lambda)^r$ in $F(x)$, taking the jth derivative of the remaining expression, and then letting $x = \lambda$ (while dividing by $j!$).

■ **Example B.10**

Expand $F(x)$ into partial fractions if

$$F(x) = \frac{4x^3 + 16x^2 + 23x + 13}{(x+1)^3(x+2)}$$

The partial fractions are

$$F(x) = \frac{a_0}{(x+1)^3} + \frac{a_1}{(x+1)^2} + \frac{a_2}{x+1} + \frac{k}{x+2}$$

The coefficient k is obtained by concealing the factor $(x+2)$ in $F(x)$ and then substituting $x = -2$ in the remaining expression:

$$k = \frac{4x^3 + 16x^2 + 23x + 13}{(x+1)^3 \text{[redacted]}}\bigg|_{x=-2} = 1$$

To find a_0, we conceal the factor $(x+1)^3$ in $F(x)$ and let $x = -1$ in the remaining expression:

$$a_0 = \frac{4x^3 + 16x^2 + 23x + 13}{\text{[redacted]}(x+2)}\bigg|_{x=-1} = 2$$

To find a_1, we conceal the factor $(x+1)^3$ in $F(x)$, take the derivative of the remaining expression, and then let $x = -1$:

$$a_1 = \frac{d}{dx}\left[\frac{4x^3+16x^2+23x+13}{\text{[redacted]}(x+2)}\right]\Bigg|_{x=-1} = 1$$

Similarly,

$$a_2 = \frac{1}{2!}\frac{d^2}{dx^2}\left[\frac{4x^3+16x^2+23x+13}{\text{[redacted]}(x+2)}\right]\Bigg|_{x=-1} = 3$$

Therefore,

$$F(x) = \frac{2}{(x+1)^3} + \frac{1}{(x+1)^2} + \frac{3}{x+1} + \frac{1}{x+2} \quad \blacksquare$$

B.5-4 A Hybrid Method: Mixture of the Heaviside "Cover-Up" and Clearing Fractions

For multiple roots, especially of higher order, the Heaviside expansion method, which requires repeated differentiation, can become cumbersome. For a function which contains several repeated and unrepeated roots, a hybrid of the two procedures proves the best. The simpler coefficients are determined by the Heaviside method, and the remaining coefficients are found by clearing fractions or short-cuts, thus incorporating the best of the two methods. We demonstrate this procedure by solving Example B.10 once again by this method.

In Example B.10, coefficients k and a_0 are relatively simple to determine by the Heaviside expansion method. These values were found to be $k_1 = 1$ and $a_0 = 2$. Therefore,

$$\frac{4x^3+16x^2+23x+13}{(x+1)^3(x+2)} = \frac{2}{(x+1)^3} + \frac{a_1}{(x+1)^2} + \frac{a_2}{x+1} + \frac{1}{x+2}$$

We now multiply both sides of the above equation by $(x+1)^3(x+2)$ to clear the fractions. This yields

$$\begin{aligned}
&4x^3+16x^2+23x+13 \\
&\quad = 2(x+2) + a_1(x+1)(x+2) + a_2(x+1)^2(x+2) + (x+1)^3 \\
&\quad = (1+a_2)x^3 + (a_1+4a_2+3)x^2 + (5+3a_1+5a_2)x + (4+2a_1+2a_2+1)
\end{aligned}$$

Equating coefficients of the third and second powers of x on both sides, we obtain

$$\left.\begin{aligned} 1+a_2 &= 4 \\ a_1+4a_2+3 &= 16 \end{aligned}\right\} \Longrightarrow \quad \begin{aligned} a_1 &= 1 \\ a_2 &= 3 \end{aligned}$$

We may stop here if we wish because the two desired coefficients, a_1 and a_2, are now determined. However, equating the coefficients of the two remaining powers of x yields a convenient check on the answer. Equating the coefficients of the x^1 and x^0 terms, we obtain

$$23 = 5 + 3a_1 + 5a_2$$

$$13 = 4 + 2a_1 + 2a_2 + 1$$

These equations are satisfied by the values $a_1 = 1$ and $a_2 = 3$, found earlier, providing an additional check for our answers. Therefore,

$$F(x) = \frac{2}{(x+1)^3} + \frac{1}{(x+1)^2} + \frac{3}{x+1} + \frac{1}{x+2}$$

which agrees with the previous result.

A Mixture of the Heaviside "Cover-Up" and Short Cuts

In the above example, after determining the coefficients $a_0 = 2$ and $k = 1$ by the Heaviside method as before, we have

$$\frac{4x^3 + 16x^2 + 23x + 13}{(x+1)^3(x+2)} = \frac{2}{(x+1)^3} + \frac{a_1}{(x+1)^2} + \frac{a_2}{x+1} + \frac{1}{x+2}$$

There are only two unknown coefficients, a_1 and a_2. If we multiply both sides of the above equation by x and then let $x \to \infty$, we can eliminate a_1. This yields

$$4 = a_2 + 1 \quad \Longrightarrow \quad a_2 = 3$$

Therefore,

$$\frac{4x^3 + 16x^2 + 23x + 13}{(x+1)^3(x+2)} = \frac{2}{(x+1)^3} + \frac{a_1}{(x+1)^2} + \frac{3}{x+1} + \frac{1}{x+2}$$

There is now only one unknown a_1, which can be readily found by setting x equal to any convenient value, say $x = 0$. This yields

$$\tfrac{13}{2} = 2 + a_1 + 3 + \tfrac{1}{2} \quad \Longrightarrow \quad a_1 = 1$$

which agrees with our earlier answer.

B.5-5 Improper $F(x)$ with $m = n$

A general method of handling an improper function is indicated in the beginning of this section. However, for a special case where the numerator and denominator polynomials of $F(x)$ are of the same degree ($m = n$), the procedure is the same as that for a proper function. We can show that for

$$\begin{aligned} F(x) &= \frac{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0}{x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0} \\ &= b_n + \frac{k_1}{x - \lambda_1} + \frac{k_2}{x - \lambda_2} + \cdots + \frac{k_n}{x - \lambda_n} \end{aligned}$$

the coefficients $k_1, k_2, \ldots, k_n$ are computed as if $F(x)$ were proper. Thus,

$$k_r = (x - \lambda_r)F(x)\big|_{x=\lambda_r}$$

For quadratic or repeated factors, the appropriate procedures discussed in Secs. B.5-2 or B.5-3 should be used as if $F(x)$ were proper. In other words, when $m = n$, the only difference between the proper and improper case is the appearance of an extra constant b_n in the latter. Otherwise the procedure remains the same. The proof is left as an exercise for the reader.

Example B.11

Expand $F(x)$ into partial fractions if

$$F(x) = \frac{3x^2 + 9x - 20}{x^2 + x - 6} = \frac{3x^2 + 9x - 20}{(x-2)(x+3)}$$

Here $m = n = 2$ with $b_n = b_2 = 3$. Therefore,

$$F(x) = \frac{3x^2 + x - 20}{(x-2)(x+3)} = 3 + \frac{k_1}{x-2} + \frac{k_2}{x+3}$$

in which

$$k_1 = \left.\frac{3x^2 + 9x - 20}{\text{[redacted]}(x+3)}\right|_{x=2} = \frac{12 + 18 - 20}{(2+3)} = \frac{10}{5} = 2$$

and

$$k_2 = \left.\frac{3x^2 + 9x - 20}{(x-2)\text{[redacted]}}\right|_{x=-3} = \frac{27 - 27 - 20}{(-3-2)} = \frac{-20}{-5} = 4$$

Therefore,

$$F(x) = \frac{3x^2 + 9x - 20}{(x-2)(x+3)} = 3 + \frac{2}{x-2} + \frac{4}{x+3}$$

B.5-6 Modified Partial Fractions

Often we require partial fractions of the form $\frac{kx}{(x-\lambda_i)^r}$ rather than $\frac{k}{(x-\lambda_i)^r}$. This can be achieved by expanding $F(x)/x$ into partial fractions. Consider, for example,

$$F(x) = \frac{5x^2 + 20x + 18}{(x+2)(x+3)^2}$$

Dividing both sides by x yields

$$\frac{F(x)}{x} = \frac{5x^2 + 20x + 18}{x(x+2)(x+3)^2}$$

Expansion of the right-hand side into partial fractions as usual yields

$$\frac{F(x)}{x} = \frac{5x^2 + 20x + 18}{x(x+2)(x+3)^2} = \frac{a_1}{x} + \frac{a_2}{x+2} + \frac{a_3}{(x+3)} + \frac{a_4}{(x+3)^2}$$

Using the procedure discussed earlier, we find $a_1 = 1$, $a_2 = 1$, $a_3 = -2$, and $a_4 = 1$. Therefore,

$$\frac{F(x)}{x} = \frac{1}{x} + \frac{1}{x+2} - \frac{2}{x+3} + \frac{1}{(x+3)^2}$$

Now multiplying both sides by x yields

$$F(x) = 1 + \frac{x}{x+2} - \frac{2x}{x+3} + \frac{x}{(x+3)^2}$$

This expresses $F(x)$ as the sum of partial fractions having the form $\frac{kx}{(x-\lambda_i)^r}$.

B.6 Vectors and Matrices

An entity specified by n numbers in a certain order (ordered n-tuple) is an n-dimensional **vector**. Thus, an ordered n-tuple $(x_1, x_2, \ldots, x_n)$ represents an n-dimensional vector $\mathbf{x}$. Vectors may be represented as a row (**row vector**):

$$\mathbf{x} = [x_1 \quad x_2 \quad \cdots \quad x_n]$$

or as a column (**column vector**):

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Simultaneous linear equations can be viewed as the transformation of one vector into another. Consider, for example, the n simultaneous linear equations

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{aligned} \tag{B.48}$$

If we define two column vectors $\mathbf{x}$ and $\mathbf{y}$ as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \tag{B.49}$$

then Eqs. (B.48) may be viewed as the relationship or the function that transforms vector $\mathbf{x}$ into vector $\mathbf{y}$. Such a transformation is called the **linear transformation** of vectors. In order to perform a linear transformation, we need to define the array of coefficients a_{ij} appearing in Eqs. (B.48). This array is called a **matrix** and is denoted by $\mathbf{A}$ for convenience:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{B.50}$$

A matrix with m rows and n columns is called a matrix of the order (m, n) or an $(m \times n)$ matrix. For the special case where $m = n$, the matrix is called a **square matrix** of order n.

It should be stressed at this point that a matrix is not a number such as a determinant, but an array of numbers arranged in a particular order. It is convenient to abbreviate the representation of matrix $\mathbf{A}$ in Eq. (B.50) with the form $(a_{ij})_{m \times n}$, implying a matrix of order $m \times n$ with a_{ij} as its ijth element. In practice, when the order $m \times n$ is understood or need not be specified, the notation can be abbreviated to (a_{ij}). Note that the first index i of a_{ij} indicates the row and the second index j indicates the column of the element a_{ij} in matrix $\mathbf{A}$.

The simultaneous equations (B.48) may now be expressed in a symbolic form as

$$\mathbf{y} = \mathbf{A}\mathbf{x} \tag{B.51}$$

or

$$\begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ y_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{bmatrix} \tag{B.52}$$

Equation (B.51) is the symbolic representation of Eq. (B.48). As yet, we have not defined the operation of the multiplication of a matrix by a vector. The quantity $\mathbf{Ax}$ is not meaningful until we define such an operation.

B.6-1 Some Definitions and Properties

A square matrix whose elements are zero everywhere except on the main diagonal is a **diagonal matrix**. An example of a diagonal matrix is

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

A diagonal matrix with unity for all its diagonal elements is called an **identity matrix** or a **unit matrix**, denoted by $\mathbf{I}$. Note that this is a square matrix:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{B.53}$$

The order of the unit matrix is sometimes indicated by a subscript. Thus, $\mathbf{I}_n$ represents the $n \times n$ unit matrix (or identity matrix). However, we shall omit the subscript. The order of the unit matrix will be understood from the context.

A matrix having all its elements zero is a **zero matrix**.

A square matrix $\mathbf{A}$ is a **symmetric matrix** if $a_{ij} = a_{ji}$ (symmetry about the main diagonal).

Two matrices of the same order are said to be **equal** if they are equal element by element. Thus, if

$$\mathbf{A} = (a_{ij})_{m\times n} \qquad \text{and} \qquad \mathbf{B} = (b_{ij})_{m\times n}$$

then $\mathbf{A} = \mathbf{B}$ only if $a_{ij} = b_{ij}$ for all i and j.

If the rows and columns of an $m \times n$ matrix $\mathbf{A}$ are interchanged so that the elements in the ith row now become the elements of the ith column (for i =1, 2, ... , m), the resulting matrix is called the **transpose** of $\mathbf{A}$ and is denoted by $\mathbf{A}^T$. It is evident that $\mathbf{A}^T$ is an $n \times m$ matrix. For example, if

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{then} \quad \mathbf{A}^T = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

Thus, if

$$\mathbf{A} = (a_{ij})_{m\times n}$$

then

$$\mathbf{A}^T = (a_{ji})_{n\times m} \tag{B.54}$$

Note that

$$(\mathbf{A}^T)^T = \mathbf{A} \tag{B.55}$$

B.6-2 Matrix Algebra

We shall now define matrix operations, such as addition, subtraction, multiplication, and division of matrices. The definitions should be formulated so that they are useful in the manipulation of matrices.

1. Addition of Matrices

For two matrices $\mathbf{A}$ and $\mathbf{B}$, both of the same order $(m \times n)$,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

we define the sum $\mathbf{A} + \mathbf{B}$ as

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} (a_{11}+b_{11}) & (a_{12}+b_{12}) & \cdots & (a_{1n}+b_{1n}) \\ (a_{21}+b_{21}) & (a_{22}+b_{22}) & \cdots & (a_{2n}+b_{2n}) \\ \cdots & \cdots & \cdots & \cdots \\ (a_{m1}+b_{m1}) & (a_{m2}+b_{m2}) & \cdots & (a_{mn}+b_{mn}) \end{bmatrix}$$

or

$$\mathbf{A}+\mathbf{B} = (a_{ij}+b_{ij})_{m\times n}$$

Note that two matrices can be added only if they are of the same order.

2. Multiplication of a Matrix by a Scalar

We define the multiplication of a matrix $\mathbf{A}$ by a scalar c as

$$c\mathbf{A} = c\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}$$

3. Matrix Multiplication

We define the product

$$\mathbf{AB} = \mathbf{C}$$

in which c_{ij}, the element of $\mathbf{C}$ in the ith row and jth column, is found by adding the products of the elements of $\mathbf{A}$ in the ith row with the corresponding elements of $\mathbf{B}$ in the jth column. Thus,

$$\begin{aligned} c_{ij} &= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} \\ &= \sum_{k=1}^{n} a_{ik}b_{kj} \end{aligned} \tag{B.56}$$

This result is shown below.

$$\underbrace{\begin{bmatrix} & & \\ & & \\ a_{i1} & a_{i2} & \cdots a_{in} \\ & & \\ & & \end{bmatrix}}_{\mathbf{A}(m\times n)} \underbrace{\begin{bmatrix} & b_{1j} & \\ & b_{2j} & \\ & \vdots & \\ \cdots & b_{ij} & \cdots \\ & \vdots & \\ & b_{nj} & \end{bmatrix}}_{\mathbf{B}(n\times p)} = \underbrace{\begin{bmatrix} & & \\ & & \\ \cdots & c_{ij} & \cdots \\ & & \\ & & \end{bmatrix}}_{\mathbf{C}(m\times p)}$$

Note carefully that the number of columns of **A** must be equal to the number of rows of **B** if this procedure is to work. In other words, **AB**, the product of matrices **A** and **B**, is defined only if the number of columns of **A** is equal to the number of rows of **B**. If this condition is not satisfied, the product **AB** is not defined and is meaningless. When the number of columns of **A** is equal to the number of rows of **B**, matrix **A** is said to be **conformable** to matrix **B** for the product **AB**. Observe that if **A** is an $m \times n$ matrix and **B** is an $n \times p$ matrix, **A** and **B** are conformable for the product, and **C** is an $m \times p$ matrix.

We demonstrate the use of the rule in Eq. (B.56) with the following examples.

$$\begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 9 & 5 & 7 \\ 3 & 4 & 2 & 3 \\ 5 & 10 & 4 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 8$$

In both cases above, the two matrices are conformable. However, if we interchange the order of the matrices as follows,

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}$$

the matrices are no longer conformable for the product. It is evident that in general,

$$\mathbf{AB} \neq \mathbf{BA}$$

Indeed, **AB** may exist and **BA** may not exist, or vice versa, as in the above examples. We shall see later that for some special matrices,

$$\mathbf{AB} = \mathbf{BA} \tag{B.57}$$

When Eq. (B.57) is true, matrices **A** and **B** are said to **commute**. We must stress here again that in general, matrices do not commute. Operation (B.57) is valid only for some special cases.

In the matrix product **AB**, matrix **A** is said to be **postmultiplied** by **B** or matrix **B** is said to be **premultiplied** by **A**. We may also verify the following relationships:

$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC} \tag{B.58}$$

$$\mathbf{C}(\mathbf{A} + \mathbf{B}) = \mathbf{CA} + \mathbf{CB} \tag{B.59}$$

We can verify that any matrix **A** premultiplied or postmultiplied by the identity matrix **I** remains unchanged:

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A} \tag{B.60}$$

Of course, we must make sure that the order of **I** is such that the matrices are conformable for the corresponding product.

4. Multiplication of a Matrix by a Vector

Consider the matrix Eq. (B.52), which represents Eq. (B.48). The right-hand side of Eq. (B.52) is a product of the $m \times n$ matrix **A** and a vector **x**. If, for the time being, we treat the vector **x** as if it were an $n \times 1$ matrix, then the product **Ax**, according to the matrix multiplication rule, yields the right-hand side of Eq. (B.48). Thus, we may multiply a matrix by a vector by treating the vector as if it were an $n \times 1$ matrix. Note that the constraint of conformability still applies. Thus, in this case, **xA** is not defined and is meaningless.

5. Matrix Inversion

To define the inverse of a matrix, let us consider the set of equations

$$\begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} \tag{B.61}$$

We can solve this set of equations for $x_1, x_2, \ldots, x_n$ in terms of $y_1, y_2, \ldots, y_n$ by using Cramer's rule [see Eq. (B.31)]. This yields

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \frac{|\mathbf{D}_{11}|}{|\mathbf{A}|} & \frac{|\mathbf{D}_{21}|}{|\mathbf{A}|} & \cdots & \frac{|\mathbf{D}_{n1}|}{|\mathbf{A}|} \\ \frac{|\mathbf{D}_{12}|}{|\mathbf{A}|} & \frac{|\mathbf{D}_{22}|}{|\mathbf{A}|} & \cdots & \frac{|\mathbf{D}_{n2}|}{|\mathbf{A}|} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{|\mathbf{D}_{1n}|}{|\mathbf{A}|} & \frac{|\mathbf{D}_{2n}|}{|\mathbf{A}|} & \cdots & \frac{|\mathbf{D}_{nn}|}{|\mathbf{A}|} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \tag{B.62}$$

in which $|\mathbf{A}|$ is the determinant of the matrix **A** and $|\mathbf{D}_{ij}|$ is the cofactor of element a_{ij} in the matrix **A**. The cofactor of element a_{ij} is given by $(-1)^{i+j}$ times the determinant of the $(n-1) \times (n-1)$ matrix that is obtained when the ith row and the jth column in matrix **A** are deleted.

We can express Eq. (B.61) in matrix form as

$$\mathbf{y} = \mathbf{Ax} \tag{B.63}$$

We can now define $\mathbf{A}^{-1}$, the inverse of a square matrix **A**, with the property

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \qquad \text{(unit matrix)} \tag{B.64}$$

Then, premultiplying both sides of Eq. (B.63) by $\mathbf{A}^{-1}$, we obtain

$$\mathbf{A}^{-1}\mathbf{y} = \mathbf{A}^{-1}\mathbf{Ax} = \mathbf{Ix} = \mathbf{x}$$

or

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{y} \tag{B.65}$$

A comparison of Eq. (B.65) with Eq. (B.62) shows that

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\begin{bmatrix} |\mathbf{D}_{11}| & |\mathbf{D}_{21}| & \cdots & |\mathbf{D}_{n1}| \\ |\mathbf{D}_{12}| & |\mathbf{D}_{22}| & \cdots & |\mathbf{D}_{n2}| \\ \cdots & \cdots & \cdots & \cdots \\ |\mathbf{D}_{1n}| & |\mathbf{D}_{2n}| & \cdots & |\mathbf{D}_{nn}| \end{bmatrix} \tag{B.66}$$

One of the conditions necessary for a unique solution of Eq. (B.61) is that the number of equations must equal the number of unknowns. This implies that the matrix $\mathbf{A}$ must be a square matrix. In addition, we observe from the solution as given in Eq. (B.62) that if the solution is to exist, $|\mathbf{A}| \neq 0$.† Therefore, the inverse exists only for a square matrix and only under the condition that the determinant of the matrix be nonzero. A matrix whose determinant is nonzero is a **nonsingular** matrix. Thus, an inverse exists only for a nonsingular (square) matrix. By definition, we have

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \tag{B.67a}$$

Postmultiplying this equation by $\mathbf{A}^{-1}$ and then premultiplying by $\mathbf{A}$, we can show that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} \tag{B.67b}$$

Note that the matrices $\mathbf{A}$ and $\mathbf{A}^{-1}$ commute.

■ **Example B.12**

Let us find $\mathbf{A}^{-1}$ if

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

Here

$$|\mathbf{D}_{11}| = -4, \quad |\mathbf{D}_{12}| = 8, \quad |\mathbf{D}_{13}| = -4$$

$$|\mathbf{D}_{21}| = 1, \quad |\mathbf{D}_{22}| = -1, \quad |\mathbf{D}_{23}| = -1$$

$$|\mathbf{D}_{31}| = 1, \quad |\mathbf{D}_{32}| = -5, \quad |\mathbf{D}_{33}| = 3$$

and $|\mathbf{A}| = -4$. Therefore,

$$\mathbf{A}^{-1} = -\frac{1}{4}\begin{bmatrix} -4 & 1 & 1 \\ 8 & -1 & -5 \\ -4 & -1 & 3 \end{bmatrix}$$ ■

†These two conditions imply that the number of equations is equal to the number of unknowns and that all the equations are independent.

B.6-3 Derivatives and Integrals of a Matrix

Elements of a matrix need not be constants; they may be functions of a variable. For example, if

$$\mathbf{A} = \begin{bmatrix} e^{-2t} & \sin t \\ e^t & e^{-t} + e^{-2t} \end{bmatrix} \tag{B.68}$$

then the matrix elements are functions of t. Here, it is helpful to denote $\mathbf{A}$ by $\mathbf{A}(t)$. Also, it would be helpful to define the derivative and integral of $\mathbf{A}(t)$.

The derivative of a matrix $\mathbf{A}(t)$ (with respect to t) is defined as a matrix whose ijth element is the derivative (with respect to t) of the ijth element of the matrix $\mathbf{A}$. Thus, if

$$\mathbf{A}(t) = [a_{ij}(t)]_{m \times n}$$

then

$$\frac{d}{dt}[\mathbf{A}(t)] = \left[\frac{d}{dt} a_{ij}(t)\right]_{m \times n} \tag{B.69a}$$

or

$$\dot{\mathbf{A}}(t) = [\dot{a}_{ij}(t)]_{m \times n} \tag{B.69b}$$

Thus, the derivative of the matrix in Eq. (B.68) is given by

$$\dot{\mathbf{A}}(t) = \begin{bmatrix} -2e^{-2t} & \cos t \\ e^t & -e^{-t} - 2e^{-2t} \end{bmatrix}$$

Similarly, we define the integral of $\mathbf{A}(t)$ (with respect to t) as a matrix whose ijth element is the integral (with respect to t) of the ijth element of the matrix $\mathbf{A}$:

$$\int \mathbf{A}(t)\, dt = \left(\int a_{ij}(t)\, dt\right)_{m \times n} \tag{B.70}$$

Thus, for the matrix $\mathbf{A}$ in Eq. (B.68), we have

$$\int \mathbf{A}(t)\, dt = \begin{bmatrix} \int e^{-2t}\, dt & \int \sin\, dt \\ \int e^t\, dt & \int (e^{-t} + 2e^{-2t})\, dt \end{bmatrix}$$

We can readily prove the following identities:

$$\frac{d}{dt}(\mathbf{A} + \mathbf{B}) = \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{B}}{dt} \tag{B.71a}$$

$$\frac{d}{dt}(c\mathbf{A}) = c\frac{d\mathbf{A}}{dt} \tag{B.71b}$$

$$\frac{d}{dt}(\mathbf{AB}) = \frac{d\mathbf{A}}{dt}\mathbf{B} + \mathbf{A}\frac{d\mathbf{B}}{dt} = \dot{\mathbf{A}}\mathbf{B} + \mathbf{A}\dot{\mathbf{B}} \tag{B.71c}$$

The proofs of identities (B.71a) and (B.71b) are trivial. We can prove Eq. (B.71c) as follows. Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{B}$ an $n \times p$ matrix. Then, if

$$\mathbf{C} = \mathbf{AB}$$

from Eq. (B.56), we have

$$c_{ik} = \sum_{j-1}^{n} a_{ij} b_{jk}$$

and

$$\dot{c}_{ik} = \underbrace{\sum_{j-1}^{n} \dot{a}_{ij} b_{jk}}_{d_{ik}} + \underbrace{\sum_{j-1}^{n} a_{ij} \dot{b}_{jk}}_{e_{ik}} \tag{B.72}$$

or

$$\dot{c}_{ij} = d_{ij} + e_{ik}$$

Equation (B.72) along with the multiplication rule clearly indicate that d_{ik} is the ikth element of matrix $\dot{\mathbf{A}}\mathbf{B}$ and e_{ik} is the ikth element of matrix $\mathbf{A}\dot{\mathbf{B}}$. Equation (B.71c) then follows.

If we let $\mathbf{B} = \mathbf{A}^{-1}$ in Eq. (B.71c), we obtain

$$\frac{d}{dt}(\mathbf{A}\mathbf{A}^{-1}) = \frac{d\mathbf{A}}{dt}\mathbf{A}^{-1} + \mathbf{A}\frac{d}{dt}\mathbf{A}^{-1}$$

But since

$$\frac{d}{dt}(\mathbf{A}\mathbf{A}^{-1}) = \frac{d}{dt}\mathbf{I} = 0$$

we have

$$\frac{d}{dt}(\mathbf{A}^{-1}) = -\mathbf{A}^{-1}\frac{d\mathbf{A}}{dt}\mathbf{A}^{-1} \tag{B.73}$$

B.6-4 The Characteristic Equation of a Matrix: The Cayley-Hamilton Theorem

For an $(n \times n)$ square matrix $\mathbf{A}$, any vector $\mathbf{x}$ ($\mathbf{x} \neq 0$) that satisfies the equation

$$\mathbf{Ax} = \lambda \mathbf{x} \tag{B.74}$$

is an **eigenvector** (or **characteristic vector**), and λ is the corresponding **eigenvalue** (or **characteristic value**) of $\mathbf{A}$. Equation (B.74) can be expressed as

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0 \tag{B.75}$$

The solution for this set of homogeneous equations exists if and only if

$$|\mathbf{A} - \lambda \mathbf{I}| = |\lambda \mathbf{I} - \mathbf{A}| = 0 \tag{B.76a}$$

or

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0 \tag{B.76b}$$

Equation (B.76a) [or (B.76b)] is known as the **characteristic equation** of the matrix **A** and can be expressed as

$$Q(\lambda) = |\lambda\mathbf{I} - \mathbf{A}| = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0\lambda^0 = 0 \tag{B.77}$$

$Q(\lambda)$ is called the **characteristic polynomial** of the matrix **A**. The n zeros of the characteristic polynomial are the eigenvalues of **A** and, corresponding to each eigenvalue, there is an eigenvector that satisfies Eq. (B.74).

The **Cayley-Hamilton theorem** states that every $n \times n$ matrix **A** satisfies its own characteristic equation. In other words, Eq. (B.77) is valid if λ is replaced by **A**:

$$\mathbf{Q}(\mathbf{A}) = \mathbf{A}^n + a_{n-1}\mathbf{A}^{n-1} + \cdots + a_1\mathbf{A} + a_0\mathbf{A}^0 = 0 \tag{B.78}$$

Functions of a Matrix

The Cayley-Hamilton theorem can be used to evaluate functions of a square matrix **A**, as shown below.

Consider a function $f(\lambda)$ in the form of an infinite power series:

$$f(\lambda) = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda_2^2 + \cdots + \cdots = \sum_{i=0}^{\infty} \alpha_i\lambda^i \tag{B.79}$$

Because λ satisfies the characteristic Eq. (B.77), we can write

$$\lambda^n = -a_{n-1}\lambda^{n-1} - a_{n-2}\lambda^{n-2} - \cdots - a_1\lambda - a_0 \tag{B.80}$$

If we multiply both sides by λ, the left-hand side is λ^{n+1}, and the right-hand side contains the terms λ^n, λ^{n-1}, ... , λ. Using Eq. (B.80), if we substitute λ^n in terms of λ^{n-1}, λ^{n-2}, ... , λ, the highest power on the right-hand side is reduced to $n-1$. Continuing in this way, we see that λ^{n+k} can be expressed in terms of λ^{n-1}, λ^{n-2}, ... , λ for any k. Hence, the infinite series on the right-hand side of Eq. (B.79) can always be expressed in terms of λ^{n-1}, λ^{n-2}, ... , λ as

$$f(\lambda) = \beta_0 + \beta_1\lambda + \beta_2\lambda^2 + \cdots + \beta_{n-1}\lambda^{n-1} \tag{B.81}$$

If we assume that there are n distinct eigenvalues λ_1, λ_2, ... , λ_n, then Eq. (B.81) holds for these n values of λ. The substitution of these values in Eq. (B.81) yields n simultaneous equations

$$\begin{bmatrix} f(\lambda_1) \\ f(\lambda_2) \\ \cdots \\ f(\lambda_n) \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \cdots \\ \beta_{n-1} \end{bmatrix} \tag{B.82a}$$

and

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \cdots \\ \beta_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} f(\lambda_1) \\ f(\lambda_2) \\ \cdots \\ f(\lambda_n) \end{bmatrix} \tag{B.82b}$$

Since $\mathbf{A}$ also satisfies Eq. (B.80), we may advance a similar argument to show that if $f(\mathbf{A})$ is a function of a square matrix $\mathbf{A}$ expressed as an infinite power series in $\mathbf{A}$, then

$$f(\mathbf{A}) = \alpha_0\mathbf{I} + \alpha_1\mathbf{A} + \alpha_2\mathbf{A}^2 + \cdots + \cdots = \sum_{i=0}^{\infty} \alpha_i\mathbf{A}^i \tag{B.83a}$$

and

$$f(\mathbf{A}) = \beta_0\mathbf{I} + \beta_1\mathbf{A} + \beta_2\mathbf{A}^2 + \cdots + \beta_{n-1}\mathbf{A}^{n-1} \tag{B.83b}$$

in which the coefficients β_is are found from Eq. (B.82b). If some of the eigenvalues are repeated (multiple roots), the results are somewhat modified.

We shall demonstrate the utility of this result with the following two examples.

B.6-5 Computation of an Exponential and a Power of a Matrix

Let us compute $e^{\mathbf{A}t}$ defined by

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \cdots + \frac{\mathbf{A}^nt^n}{n!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{\mathbf{A}^kt^k}{k!} \end{aligned}$$

From Eq. (B.83b), we can express

$$e^{\mathbf{A}t} = \sum_{i=1}^{n-1} \beta_i(\mathbf{A})^i$$

in which the β_is are given by Eq. (B.82b), with $f(\lambda_i) = e^{\lambda_i t}$.

■ **Example B.13**

Let us consider the case where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

The eigenvalues are

$$|\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{vmatrix} = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2) = 0$$

Hence, $\lambda_1 = -1$, $\lambda_2 = -2$, and

$$e^{\mathbf{A}t} = \beta_0\mathbf{I} + \beta_1\mathbf{A}$$

in which

$$\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} e^{-t} \\ e^{-2t} \end{bmatrix}$$
$$= \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} \\ e^{-2t} \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

and

$$e^{\mathbf{A}t} = (2e^{-t} - e^{-2t}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (e^{-t} - e^{-2t}) \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$
$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & (e^{-t} - e^{-2t}) \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \tag{B.84}$$

■

Computation of $\mathbf{A}^k$

As Eq. (B.83b) indicates, we can express $\mathbf{A}^k$ as

$$\mathbf{A}^k = \beta_0\mathbf{I} + \beta_1\mathbf{A} + \cdots + \beta_{n-1}\mathbf{A}^{n-1}$$

in which the β_is are given by Eq. (B.82b) with $f(\lambda_i) = \lambda_i^k$. For a completed example of the computation of $\mathbf{A}^k$ by this method, see Example 13.12.

B.7 Miscellaneous

B.7-1 L'Hôpital's Rule

If $\lim f(x)/g(x)$ results in the indeterministic form 0/0 or ∞/∞, then

$$\lim \frac{f(x)}{g(x)} = \lim \frac{\dot{f}(x)}{\dot{g}(x)}$$

B.7-2 The Taylor and Maclaurin Series

$$f(x) = f(a) + \frac{(x-a)}{1!}\dot{f}(a) + \frac{(x-a)^2}{2!}\ddot{f}(a) + \cdots$$

$$f(x) = f(0) + \frac{x}{1!}\dot{f}(0) + \frac{x^2}{2!}\ddot{f}(0) + \cdots$$

B.7-3 Power Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots \qquad x^2 < \pi^2/4$$

$$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots \qquad x^2 < \pi^2/4$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \binom{n}{k}x^k + \cdots + x^n$$

$$\approx 1 + nx \qquad |x| \ll 1$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad |x| < 1$$

B.7-4 Sums

$$\sum_{m=0}^{k} r^m = \frac{r^{k+1} - 1}{r - 1} \qquad r \neq 1$$

$$\sum_{m=M}^{N} r^m = \frac{r^{N+1} - r^M}{r - 1} \qquad r \neq 1$$

$$\sum_{m=0}^{k} \left(\frac{a}{b}\right)^m = \frac{a^{k+1} - b^{k+1}}{b^k(a - b)} \qquad a \neq b$$

B.7-5 Complex Numbers

$$e^{\pm j\pi/2} = \pm j$$

$$e^{\pm jn\pi} = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$$

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

$$a + jb = re^{j\theta} \qquad r = \sqrt{a^2+b^2}, \qquad \theta = \tan^{-1}\left(\tfrac{b}{a}\right)$$

$$(re^{j\theta})^k = r^k e^{jk\theta}$$

$$(r_1e^{j\theta_1})(r_2e^{j\theta_2}) = r_1r_2e^{j(\theta_1+\theta_2)}$$

B.7-6 Trigonometric Identities

$$e^{\pm jx} = \cos x \pm j\sin x$$

$$\cos x = \tfrac{1}{2}[e^{jx} + e^{-jx}]$$

$$\sin x = \tfrac{1}{2j}[e^{jx} - e^{-jx}]$$

$$\cos\left(x \pm \tfrac{\pi}{2}\right) = \mp\sin x$$

$$\sin\left(x \pm \tfrac{\pi}{2}\right) = \pm\cos x$$

$$2\sin x\cos x = \sin 2x$$

$$\sin^2 x + \cos^2 x = 1$$

$$\cos^2 x - \sin^2 x = \cos 2x$$

$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$$

$$\cos^3 x = \tfrac{1}{4}(3\cos x + \cos 3x)$$

$$\sin^3 x = \tfrac{1}{4}(3\sin x - \sin 3x)$$

$$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$$

$$\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x\tan y}$$

$$\sin x\sin y = \tfrac{1}{2}[\cos(x-y) - \cos(x+y)]$$

$$\cos x\cos y = \tfrac{1}{2}[\cos(x-y) + \cos(x+y)]$$

$$\sin x\cos y = \tfrac{1}{2}[\sin(x-y) + \sin(x+y)]$$

$$a\cos x + b\sin x = C\cos(x+\theta)$$

in which $C = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}\left(\frac{-b}{a}\right)$

B.7-7 Indefinite Integrals

$$\int u\,dv = uv - \int v\,du$$

$$\int f(x)\dot{g}(x)\,dx = f(x)g(x) - \int \dot{f}(x)g(x)\,dx$$

$$\int \sin\,ax\,dx = -\frac{1}{a}\cos\,ax \qquad \int \cos\,ax\,dx = \frac{1}{a}\sin\,ax$$

$$\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin\,2ax}{4a} \qquad \int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin\,2ax}{4a}$$

$$\int x\sin\,ax\,dx = \frac{1}{a^2}(\sin\,ax - ax\cos\,ax)$$

$$\int x\cos\,ax\,dx = \frac{1}{a^2}(\cos\,ax + ax\sin\,ax)$$

$$\int x^2\sin\,ax\,dx = \frac{1}{a^3}(2ax\sin\,ax + 2\cos\,ax - a^2x^2\cos\,ax)$$

$$\int x^2\cos\,ax\,dx = \frac{1}{a^3}(2ax\cos\,ax - 2\sin\,ax + a^2x^2\sin\,ax)$$

$$\int \sin\,ax\sin\,bx\,dx = \frac{\sin\,(a-b)x}{2(a-b)} - \frac{\sin\,(a+b)x}{2(a+b)} \qquad a^2 \neq b^2$$

$$\int \sin\,ax\cos\,bx\,dx = -\left[\frac{\cos\,(a-b)x}{2(a-b)} + \frac{\cos\,(a+b)x}{2(a+b)}\right] \qquad a^2 \neq b^2$$

$$\int \cos\,ax\cos\,bx\,dx = \frac{\sin\,(a-b)x}{2(a-b)} + \frac{\sin\,(a+b)x}{2(a+b)} \qquad a^2 \neq b^2$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}$$

$$\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax - 1)$$

$$\int x^2e^{ax}\,dx = \frac{e^{ax}}{a^3}(a^2x^2 - 2ax + 2)$$

$$\int e^{ax}\sin\,bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin\,bx - b\cos\,bx)$$

$$\int e^{ax}\cos\,bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos\,bx + b\sin\,bx)$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{x}{x^2+a^2}\,dx = \frac{1}{2}\ln(x^2+a^2)$$

B.7-8 Differentiation Table

$\frac{d}{dx}f(u) = \frac{d}{du}f(u)\frac{du}{dx}$	$\frac{d}{dx}a^{bx} = b(\ln a)a^{bx}$
$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	$\frac{d}{dx}\sin ax = a\cos ax$
$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$	$\frac{d}{dx}\cos ax = -a\sin ax$
$\frac{dx^n}{dx} = nx^{n-1}$	$\frac{d}{dx}\tan ax = \frac{a}{\cos^2 ax}$
$\frac{d}{dx}\ln(ax) = \frac{1}{x}$	$\frac{d}{dx}(\sin^{-1} ax) = \frac{a}{\sqrt{1-a^2x^2}}$
$\frac{d}{dx}\log(ax) = \frac{\log e}{x}$	$\frac{d}{dx}(\cos^{-1} ax) = \frac{-a}{\sqrt{1-a^2x^2}}$
$\frac{d}{dx}e^{bx} = be^{bx}$	$\frac{d}{dx}(\tan^{-1} ax) = \frac{a}{1+a^2x^2}$

B.7-9 Some Useful Constants

$\pi \approx 3.1415926535$

$e \approx 2.7182818284$

$\frac{1}{e} \approx 0.3678794411$

$\log_{10} 2 = 0.30103$

$\log_{10} 3 = 0.47712$

B.7-10 Solution of Quadratic and Cubic Equations

Any **quadratic** equation can be reduced to the form

$$ax^2 + bx + c = 0$$

The solution of this equation is provided by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

A general **cubic** equation

$$y^3 + py^2 + qy + r = 0$$

may be reduced to the **depressed cubic** form

$$x^3 + ax + b = 0$$

by substituting

$$y = x - \frac{p}{3}$$

This yields

$$a = \frac{1}{3}(3q - p^2) \qquad b = \frac{1}{27}(2p^3 - 9pq + 27r)$$

Now let

$$A = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}, \qquad B = \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$$

The solution of the depressed cubic is

$$x = A + B, \qquad x = -\frac{A+B}{2} + \frac{A-B}{2}\sqrt{-3}, \qquad x = -\frac{A+B}{2} - \frac{A-B}{2}\sqrt{-3}$$

and

$$y = x - \frac{p}{3}$$

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